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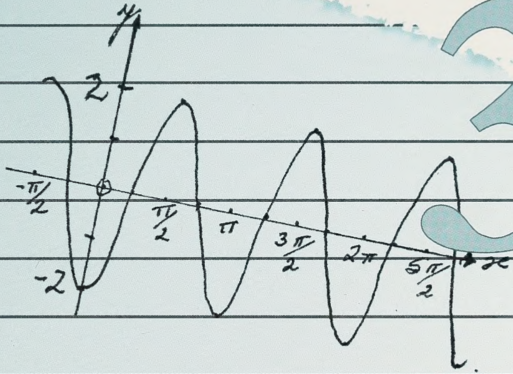
Module 3

Pure MATHEMATICS 30

Sequences and Series



$y = 2 \sin 2(x - \pi/4)$

When $x = \pi/2$,

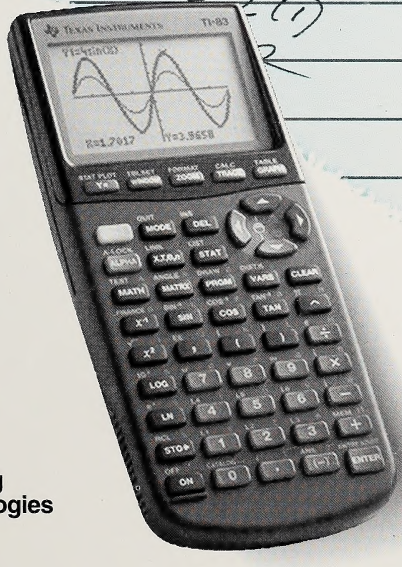
$y = 2 \sin 2(\pi/2 - \pi/4)$

$= 2 \sin 2(\pi/4)$

$= 2 \sin \pi/2$

$= 2(1)$

$= 2$



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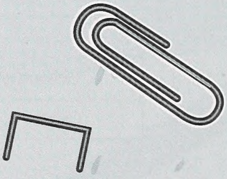
Pure MATHEMATICS 30

Module 3

Sequences and Series



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Pure Mathematics 30
Module 3: Sequences and Series
Student Module Booklet
Learning Technologies Branch
ISBN 0-7741-1655-2

This document is intended for	
Students	✓
Teachers	✓
Administrators	
Parents	
General Public	
Other	

The Learning Technologies Branch has an Internet site that you may find useful. The address is as follows:

http://www.learning.gov.ab.ca/ltb

The use of the Internet is optional. Exploring the electronic information superhighway can be educational and entertaining. However, be aware that these computer networks are not censored. Students may unintentionally or purposely find articles on the Internet that may be offensive or inappropriate. As well, the sources of information are not always cited and the content may not be accurate. Therefore, students may wish to confirm facts with a second source.

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Welcome

Pure Mathematics 30

Module 1
Transformations

Module 2
Exponents and Logarithms

Module 3
Sequences and Series

Module 4
Conics

Module 5
Trigonometry

Module 6
Combinatorics

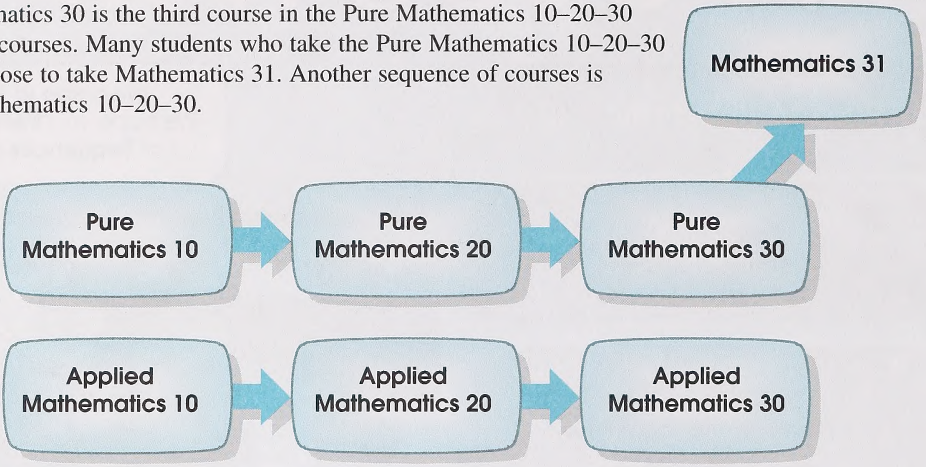
Module 7
Probability and Statistics

Welcome to **Module 3**.
We hope you'll enjoy your study of **Sequences and Series**.

Pure Mathematics 30 contains seven modules. Work through the modules in the order given, since several concepts build on each other as you progress in the course.

High School Mathematics Courses

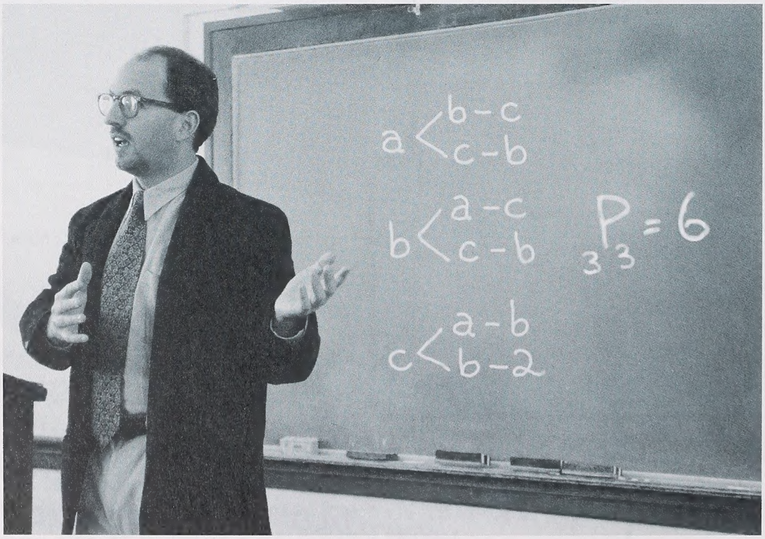
Pure Mathematics 30 is the third course in the Pure Mathematics 10–20–30 sequence of courses. Many students who take the Pure Mathematics 10–20–30 will also choose to take Mathematics 31. Another sequence of courses is Applied Mathematics 10–20–30.



How Do the Sequences Differ?

Each sequence of courses is designed for students with different mathematical strengths and interests. The Pure Mathematics 10–20–30 sequence is intended for students who are strong in mathematical theory and algebra. The Applied Mathematics 10–20–30 sequence is better suited to students who prefer to solve mathematical problems using numerical reasoning or geometry.

Each sequence of courses is designed for students with different career plans. Students who plan to take courses such as commerce, engineering, physics, and engineering technologies in post-secondary education will need to take Pure Mathematics 30 in high school. Some of these students will also need to take Mathematics 31.



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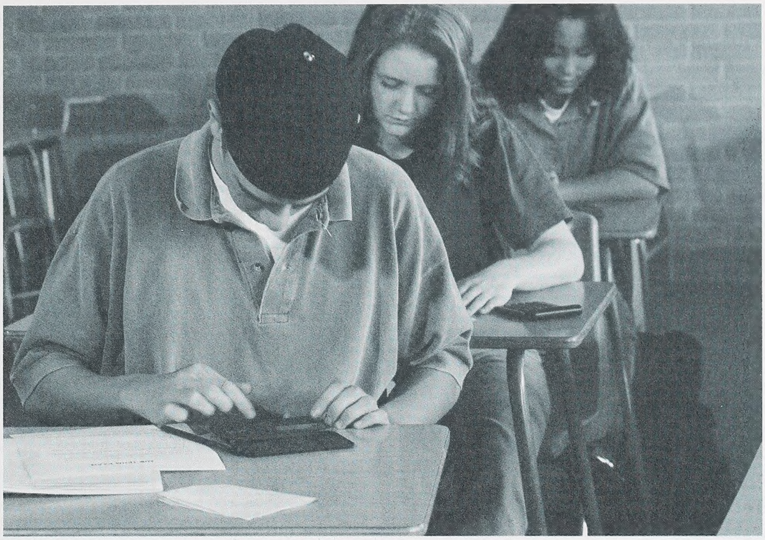
Where Can I Obtain Diploma Examination Information?



PHOTODISC, INC.

Alberta Students will write a diploma examination at the end of the course. Alberta Learning provides several documents to help students prepare for this examination. These documents are found in the directory under "Students and Learning" on the Alberta Learning website, **http://www.learning.gov.ab.ca**. Information like course expectations, the makeup of the diploma examination, keyed copies of previous examinations, preparation guides, and calculator policies are available to students at this site.

Each year, in February and September, Alberta Learning provides teachers with information on a **student project**, which teachers **may** use as part of your overall assessment. Information to students will also be posted on the Alberta Learning website. Check with your teacher to determine what you will be expected to do. Be aware that one of the diploma examination's written-response questions is worth 10% of your diploma examination mark and will deal with elements of this project.

You should take advantage of the many sources of information about Pure Mathematics 30. Your success depends on your understanding of course expectations and evaluation procedures. Work closely with your teacher and do not hesitate to ask questions.

Remember, take the initiative to find out all you can about Pure Mathematics 30.

Can I Transfer to the Applied Mathematics Sequence of Courses?

If you do not successfully complete Pure Mathematics 30, you can repeat the course or you can transfer to the Applied Mathematics sequence. If you decide to transfer to the Applied Mathematics sequence, you may take the five-credit course called Applied Mathematics 20b or you may take the five-credit course called Applied Mathematics 20.

Mathematical Processes

Throughout the course you will be expected to do the following:

- communicate mathematically
- connect mathematical ideas to everyday experiences and to concepts in other disciplines
- develop and use estimation strategies
- develop and use mental math strategies
- develop and use problem-solving strategies
- reason and justify your thinking
- select and use appropriate technologies as tools to solve problems
- use visualization to assist in processing information, making connections, and solving problems

Mathematical Process

■ Communication
■ Connection
■ Estimation
■ Mental Math
■ Problem Solving
■ Reasoning
■ Technology
■ Visualization

This icon will show which mathematical process you are expected to perform.

In order to develop these mathematical processes more fully, you are encouraged to ask someone who is also taking Pure Mathematics 30 to be your study partner. You may find that having a friend to discuss mathematics with makes your studying more enjoyable.

You are also encouraged to record your mathematical ideas in a journal. Here are some suggestions for organizing your journal:

- Keep a section on new concepts and procedures you have learned. Get in the habit of describing new concepts, procedures, and strategies in your own words. If you are having difficulty, write your questions in your journal before discussing them with your teacher or study partner. Record useful ways to help you remember what a concept means. Make graphic organizers (such as context webs, Venn diagrams, or organizational charts) to help you connect mathematical ideas.

- Keep a section on your mathematical accomplishments. This can include solutions to problems that you are proud of solving. It can also include landmark events, such as when you finally grasped a difficult concept (an "aha!") or when you first used a calculator or spreadsheet program in a new way.

- Keep a section on mathematics in the everyday world. This could include newspaper articles, cartoons, and descriptions of careers that require a mathematical background.

Resources You Will Need

In order to complete Pure Mathematics 30 successfully, you will need to have access to the following resources, which can be purchased through the Learning Resources Distributing Centre:

- the *MATHPOWER™ 12* textbook, Western Edition, published by McGraw-Hill Ryerson (2000)
- a graphing calculator

  Any of the following graphing calculators may be used: *Texas Instruments* TI-83, TI-83 Plus, TI-82, TI-86, TI-89, TI-92, or TI-92 Plus; *Casio* CFX-9850Ga-Plus, CFX-9850G, CFX-9800G, or FX-9700 series; or *Sharp* EL-9600c, EL-9600, EL-9300, or EL-9200. **Note:** Some of these calculators are no longer commercially manufactured.

- a computer and a spreadsheet program

  Any of the following spreadsheet programs may be used: *ClarisWorks™ 4.0* (or higher) for Macintosh, *ClarisWorks™ 4.0* (or higher) for Windows, *Microsoft® Works 3.0* (or higher) for Windows, or *Microsoft® Access 97* and *Microsoft® Excel 97* for Windows (a part of *Microsoft® Office Professional 95* or *Microsoft® Office Professional 97*).

Note: Examples in this course show the TI-83 graphing calculator and the *Microsoft® Excel 97* spreadsheet program where applicable. If you have access to a videocassette recorder, you may wish to view the video *The TI-83 Graphing Calculator Video Tutor* to review some of the calculator's features. Check with your distance learning provider concerning the availability of this video.

Finally, you will need the following resources, which can be purchased locally:

- a mathematical instrument set (compass, protractor, ruler, and triangles)
- a notebook or binder in which to respond to the questions asked
- a notebook or binder for journal writing

Other Visual Cues

In addition to the Mathematical Process icon described earlier, you will find other visual cues throughout this course. Read the explanations given to discover what each icon prompts you to do.

Refer to the textbook.

View a videocassette.

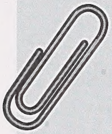
Use the Internet to explore a topic.

Answer the questions in the Assignment Booklet.

Use the Pure Mathematics 30 Companion CD.

Remember: *Any Internet website address given in this module is subject to change.*

CONTENTS



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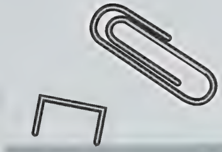

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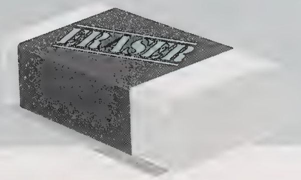
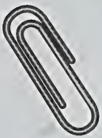
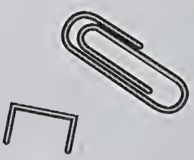






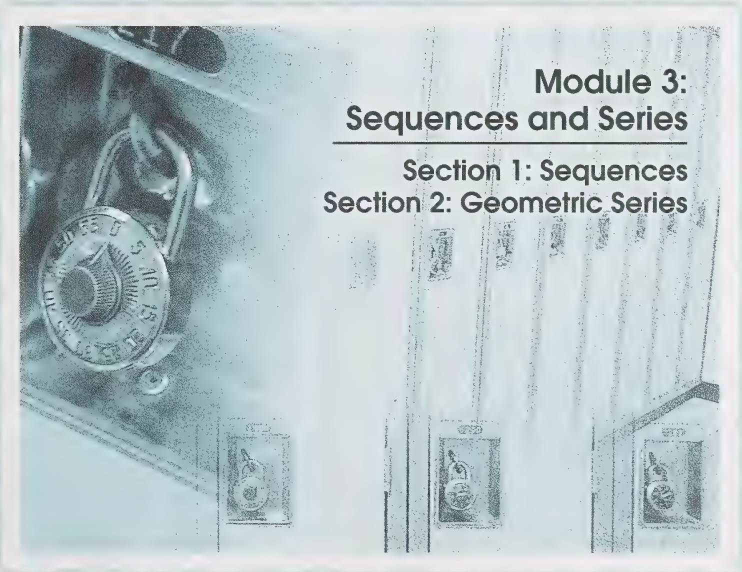
Module Overview

To open a combination lock, you have to know the three numbers that will open the lock and the correct sequence of these numbers (e.g., right 18, left 40, right 26). The advantage of a combination lock is that you never have to carry a key around with you. The disadvantage is that you may forget either the numbers in the combination or the correct sequence, especially if you don't use it regularly. Has this ever happened to you?

People perform many tasks every day that must be done sequentially. For example, a person generally purchases a January bus pass before purchasing a February pass. These monthly passes form a sequence (January, February, March, …). To determine the total amount paid for these bus passes, you would find the sum of the terms of this sequence.

In Section 1, you will examine sequences. In particular, you will investigate geometric sequences and their relationship to exponential growth. You will then use your knowledge from Module 2: Exponents and Logarithms to solve problems involving geometric sequences.

In Section 2, you will explore geometric series. These series will be obtained by adding the terms of both finite and infinite geometric sequences. Again, you will solve problems related to exponential growth, using exponents and logarithms in your exploration.



Evaluation

The document you are presently reading is called a Student Module Booklet. It will show you, step by step, what to do and how to do it.

This module, Sequences and Series, has two sections. Within each section, your work is grouped into activities. Within the activities, there are readings and questions for you to answer. By completing these questions you will construct your own learning, discover mathematical connections, and apply what you have learned. The suggested answers in the Appendix of this Student Module Booklet will provide you with immediate feedback on your progress.

In this module, you will be directed to the accompanying Assignment Booklets. You are expected to complete two Assignment Booklets for this module. Your grading in this module is based upon the assignments you submit for evaluation. The mark distribution is as follows:

Assignment Booklet 3A	
Section 1 Assignment	**30 marks**
Assignment Booklet 3B	
Section 2 Assignment	**30 marks**
Final Module Assignment	**40 marks**
TOTAL	**100 marks**

Strategies for Completing a Module

Before you begin, organize your materials: Student Module Booklet, textbook, Assignment Booklets, binder, lined paper, graph paper, graphing calculator, pens, pencils, and so on. Make sure you have a quiet area in which to work, away from distractions. Set up your time schedule.

To achieve success in this module, be sure to read all your instructions carefully and work slowly and systematically through the material. Remember, it's the work you do in this Student Module Booklet that will prepare you for your assignments and final test. Try to set realistic goals for yourself each day; and once you've set them, stick to them. Submit your assignments regularly, and don't forget to review your work before handing it in. Careful work habits will greatly increase your chances for success in Pure Mathematics 30.

Good luck!

Section 1

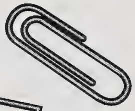
ANNIE GRIFFITHS BELT/CORBIS

Sequences

The steam engine has all but disappeared from the prairie landscape. There are still a few steam engines operating in Canada today, but those are mainly in the tourist industry as a tribute to their contribution to building a nation.

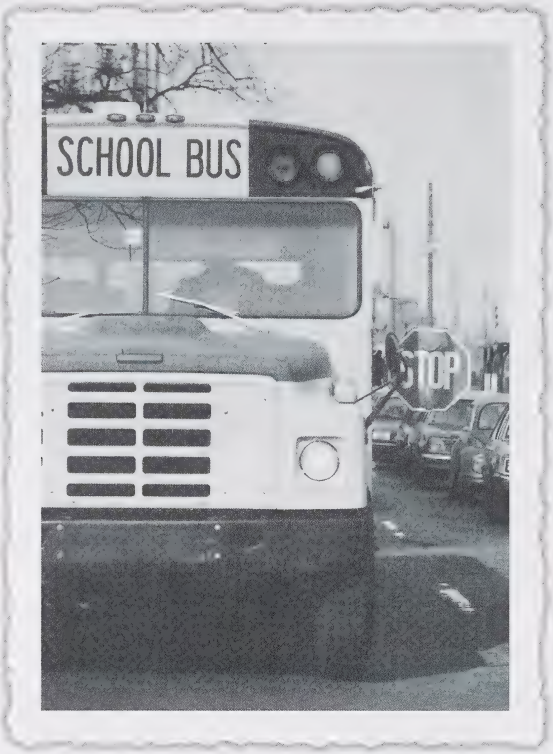
The railroad played an essential role in opening Western Canada to European settlement and to agriculture. If you study the census figures from Confederation to the present, you will see that Canada's population underwent an approximate tenfold increase. For part of Canada's history, its population was experiencing exponential growth. However, as nations mature—and Canada is no exception—population growth slows and numbers stabilize.

In this section, you will explore sequences—geometric sequences in particular. As part of your exploration, you will investigate how geometric sequences are related to exponential functions used to model growth, such as population growth and the growth of an investment bearing compound interest.



Activity 1: Sequences

If you catch a bus in the morning for school or work, you probably follow a fixed routine. From the time your alarm rings to the moment your bus arrives, you likely do things in the same order and in the same way from day to day. It is when events disrupt this schedule that you end up running late.



A sequence is a set or list of elements arranged in a particular order. In mathematics, sequences generally involve numbers.

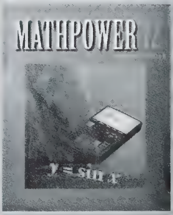
In this activity, you will explore a variety of sequences. You will explore patterns and determine expressions that represent sequences. You will then be given functions that describe sequences and be asked to find specific information about these sequences.

Mathematical Process

- ☐ Communication
- ☒ Connection
- ☐ Estimation
- ☐ Mental Math
- ☐ Problem Solving
- ☐ Reasoning
- ☐ Technology
- ☐ Visualization

a set or list of numbers arranged in a definite order

Turn to page 286 of *MATHPOWER*™*12* and read “Sequences.”



1. Answer the following questions on page 286 of the textbook:

- **a.** “Explore: Use a Pattern”
- **b.** questions 1 to 3 of “Inquire”

Turn to page 61 to compare your responses with those in the Appendix.

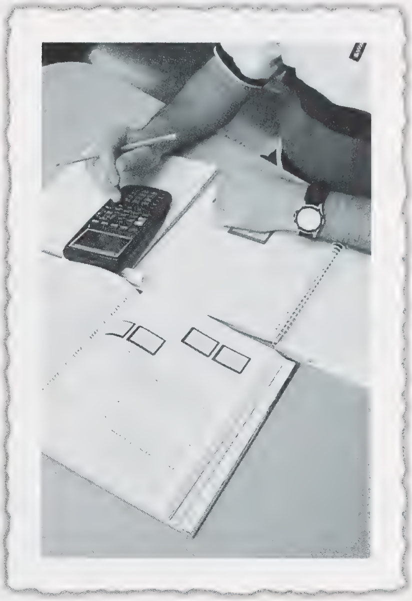
Mathematical Process

- ☐ Communication
- ☐ Connection
- ☐ Estimation
- ☐ Mental Math
- ☐ Problem Solving
- ☒ Reasoning
- ☐ Technology
- ☐ Visualization

[illegible] a formula for the nth term of a sequence, written as an expression of n

A sequence can be written as an ordered list of numbers with commas between the terms. For example, the first five odd numbers can be written as the sequence 1, 3, 5, 7, 9. Refer to the first term as t_1, the second term as t_2, and the general term as t_n. It is also common to represent the first term with a.

The general term is a rule or, more specifically, a function that describes the relationship between n, which is the term number, and the value of the term, t_n. For example, the formula for the general term might be $t_n = 2n - 1$. A formula of this type is referred to as an explicit formula.



Since n is the independent variable, it is the domain. Because the position of a term in a sequence cannot be described by a fraction, the domain is a subset of the natural numbers.

When graphing calculators are used to either list the terms of a sequence or graph the sequence, functional notation is used. For example, a sequence might be defined by $u(n) = 2n - 1$, which generates the odd numbers $\{1, 3, 5, 7, \ldots\}$. Therefore, the range is a subset of the real numbers.

Example

Determine t_5 if $t_n = 3n + 4$.

Solution

You are asked for the fifth term. Substitute 5 for n.

$$\begin{aligned} t_n &= 3n + 4 \\ t_5 &= 3(5) + 4 \\ &= 19 \end{aligned}$$

If this sequence is plotted, one point will have the coordinates of $(5, 19)$.

The fifth term of this sequence is 19.

Example

One member of the sequence $t_n = 5n - 11$ is 64. Determine which term has this value.

Solution

This means $t_n = 64$. Solve for *n*.

$$
\begin{aligned}
t_n &= 5n - 11 \\
64 &= 5n - 11 \\
75 &= 5n \\
15 &= n
\end{aligned}
$$

The 15th term is 64.

If *n* is not a member of the natural numbers, you have made an error! Check your algebra.

Sequences may also be defined by recursive formulas. For example, $t_n = t_{n-1} + n$, where $t_1 = 5$, is a recursive formula.

a formula used to define terms of a sequence by relating them to previous terms of the sequence

A recursive formula that generates a sequence is dependent on one or more previous terms. It is not defined **explicitly** in terms of *n*. Since it is dependent on at least one previous term, at least one of the values of the terms must be stated. If t_n is the *n*th term, then the previous term is t_{n-1}.

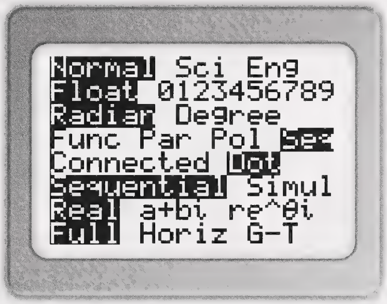
Turn to page 286 of *MATHPOWER™ 12* and read from the red line to the red line on page 288, working through Examples 1 and 2.

If you need help creating a table and graphing the sequences in Example 2 on your graphing calculator, the following information will be beneficial.

Mathematical Process

- ■ Communication
- ■ Connection
- ■ Estimation
- ■ Mental Math
- ■ Problem Solving
- ■ Reasoning
- ☒ Technology
- ☒ Visualization

Because you are working with sequences, set your calculator to Seq mode and Dot mode (as shown).



```
Normal Sci Eng
Float 0123456789
Radian Degree
Func Par Pol Seq
Connected Dot
Sequential Simul
Real a+bi re^θi
Full Horiz G-T
```

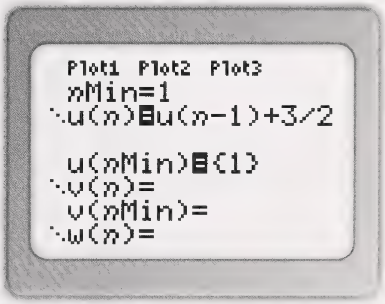
To generate the table of values for a sequence or to graph the sequence, you need to enter the function, which is the general term or explicit formula.

To enter the function $t_n = t_{n-1} + \frac{3}{2}$, where $t_1 = 1$, press the following keystrokes:

Y= 2nd [u] (X,T,Θ,*n* − 1) + 3 ÷ 2 ENTER

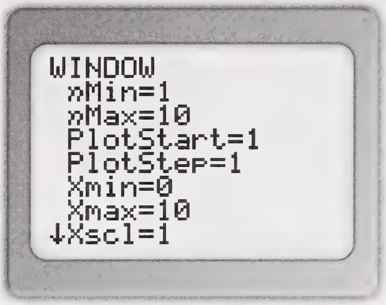
Since $t_1 = 1$, enter the minimum value for u(*n*Min) as 1. Press the following keystrokes:

1 ENTER

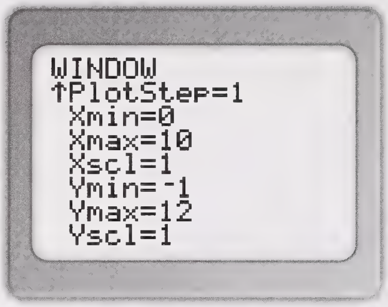


```
Plot1 Plot2 Plot3
nMin=1
⋱u(n)=u(n-1)+3/2
 u(nMin)={1}
⋱v(n)=
 v(nMin)=
⋱w(n)=
```

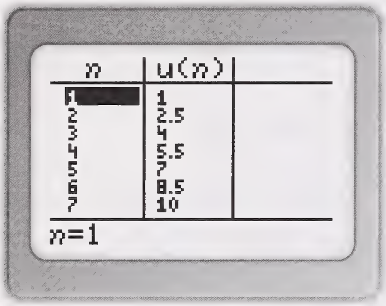
Press GRAPH to see the points of the sequence plotted. If there is no graph, you will have to adjust the window settings (domain and range). You may want to check the table of values to determine a suitable domain and range. To do this, press 2nd [TABLE].



```
WINDOW
 nMin=1
 nMax=10
 PlotStart=1
 PlotStep=1
 Xmin=0
 Xmax=10
↓Xscl=1
```

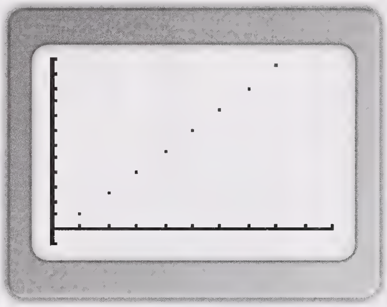


```
WINDOW
↑PlotStep=1
 Xmin=0
 Xmax=10
 Xscl=1
 Ymin=-1
 Ymax=12
 Yscl=1
```



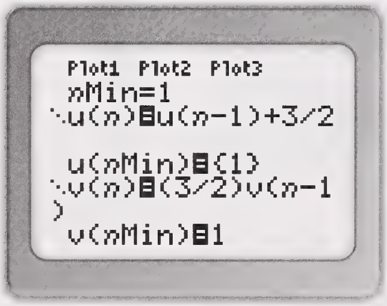
n	u(n)
1	1
2	2.5
3	4
4	5.5
5	7
6	8.5
7	10

n=1



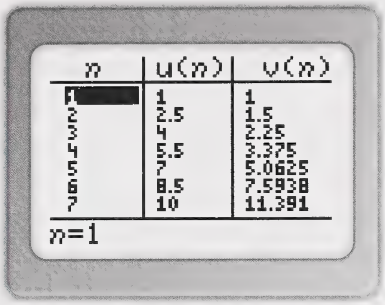
To compare the two sequences in Example 2, let $t_n = \frac{3}{2}t_n - 1$, where $t_1 = 1$, be the function represented by $v(n)$. Press Y=, select "$v(n)$," and enter the following:

(3 ÷ 2) 2nd [v] (X,T,Θ,*n* − 1) ENTER 1



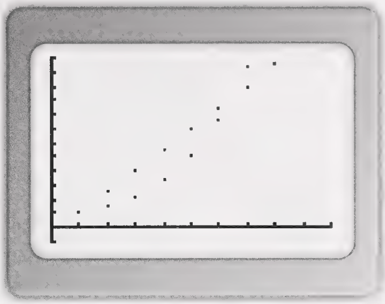
```
 Plot1 Plot2 Plot3
 nMin=1
:.u(n)=u(n-1)+3/2

 u(nMin)={1}
:.v(n)=(3/2)v(n-1
)
 v(nMin)=1
```

Now, you can display the table by pressing 2nd [TABLE] or you can display the graph of both sequences by pressing GRAPH. To compare and contrast, you may want to change the window settings.



n	u(n)	v(n)
1	1	1
2	2.5	1.5
3	4	2.25
4	5.5	3.375
5	7	5.0625
6	8.5	7.5938
7	10	11.391

n=1



In both of these examples, the first term is 1. This is not always the case.

arithmetic sequence: a sequence in which the difference between successive terms is constant

Mathematical Process

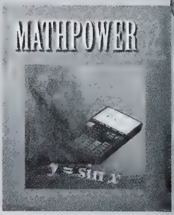
- ■ Communication
- ■ Connection
- ■ Estimation
- ■ Mental Math
- ☒ Problem Solving
- ■ Reasoning
- ■ Technology
- ■ Visualization

geometric sequence: a sequence in which the ratio of successive terms is constant

If you are given a finite set of terms, it is sometimes possible to determine the pattern. To find the pattern, you may want to start by checking to see if the same number has been added to each consecutive term. One such example is the sequence 4, 10, 16, 22, 28. It is an example of an arithmetic sequence. Notice that 6 has been added to the first term to obtain the second, that 6 has been added to the second term to obtain the third term, and so on.

If the sequence is not arithmetic, you may want to check to see if each term is the result of multiplying the previous term by a constant number. One such sequence is 4, 8, 16, 32, 64. This is an example of a geometric sequence. Each term has been multiplied by 2 to obtain the proceeding term.

If a sequence is neither arithmetic nor geometric, you have to start looking for other patterns. Try rewriting the sequence using the previous term or the previous two terms; try looking for perfect squares or perfect cubes. It is sometimes helpful to try and find a common base.



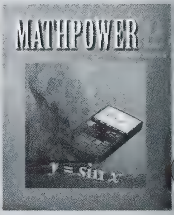
Turn to page 289 of *MATHPOWER™ 12* and work through Example 4.

2. Answer the following questions on pages 290 and 291 of the textbook:

- **a.** questions 3, 13, 14, 16, 19, 20, and 21 of "Practice"
- **b.** question 26 of "Applications and Problem Solving"

Turn to page 62 to compare your responses with those in the Appendix.

You have seen that some sequences have a common difference between consecutive terms. These sequences are called arithmetic sequences.



Turn to page 292 of *MATHPOWER™ 12* and read "Reviewing Arithmetic Sequences and Series."

3. Answer the following questions on page 292 of the textbook:

- **a.** "Explore: Use a Pattern"
- **b.** questions 1 to 4 of "Inquire"

Turn to page 70 to compare your responses with those in the Appendix.

The general rule for any arithmetic sequence is $t_n = a + (n-1)d$, where *a* is the first term, *n* is the number of terms, and *d* is the common difference between any two consecutive terms. If this sequence is plotted, the points of the sequence will lie on a straight line.

Turn to page 292 of *MATHPOWER™ 12* and read from the red line to the red line on page 293, working through Example 1.

4. Answer the following questions on pages 295 and 296 of the textbook:

- **a.** questions 3, 5, 11, 12, 18, and 20 of "Practice"
- **b.** questions 33, 35.a., 36.a., 40, and 41 of "Applications and Problem Solving"

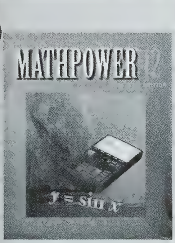
Turn to page 71 to compare your responses with those in the Appendix.

Now Try This

Mathematical Process

☐ Communication
☐ Connection
☐ Estimation
☐ Mental Math
☒ Problem Solving
☐ Reasoning
☐ Technology
☐ Visualization

In this activity, you determined an explicit formula to describe terms of a sequence. In some occasions, the pattern describing the terms of a sequence may have been challenging to find. Often, organizing information in a table will help you discover the pattern.



5. Turn to page 291 of *MATHPOWER™ 12* and answer question 29 of "Applications and Problem Solving."

Turn to page 80 to compare your responses with those in the Appendix.

Looking Back

In this activity, you explored sequences. You worked with recursive sequences and arithmetic sequences. You discovered that arithmetic sequences can be expressed using an explicit formula for the general term or defined recursively. You then determined functions that describe a finite set of numbers and used the general term or your graphing calculator to graph the sequence.

Before starting the next activity, draw the graph of $y = 5x + 3$ in your journal. If the points of a sequence, (n, t_n), where $n \in N$, lie on the line $y = 5x + 3$, determine the general term, t_n. Compare the graph of the sequence to the graph of the line, and compare the equation for t_n to the equation of the line.

What comparisons can you draw between the variables in the general term of an arithmetic sequence, $t_n = a + (n-1)d$, and the variables in the equation of a line, $y = mx + b$? Write your comparisons in your journal.

Mathematical Process

☒ Communication
☐ Connection
☐ Estimation
☐ Mental Math
☐ Problem Solving
☐ Reasoning
☐ Technology
☐ Visualization

Activity 2: Geometric Sequences

Mathematical Process

- ■ Communication
- ☒ Connection
- ■ Estimation
- ■ Mental Math
- ■ Problem Solving
- ■ Reasoning
- ■ Technology
- ■ Visualization

Have you done a biology experiment where you recorded the number of micro-organisms that grew in pond water over a given time period? If you plotted this data, where time is the independent variable and the number of micro-organisms is the dependent variable, you may have seen a set of points that look like they lie on an exponential curve.

If you counted the number of micro-organisms at regular time intervals, the counts will form a **geometric sequence**.

In this activity, you will explore geometric sequences. You will determine the general terms that describe geometric sequences, and you will determine specific information about a geometric sequence given the general term. You will also explore the connection between geometric sequences and exponential functions that describe growth or decay.

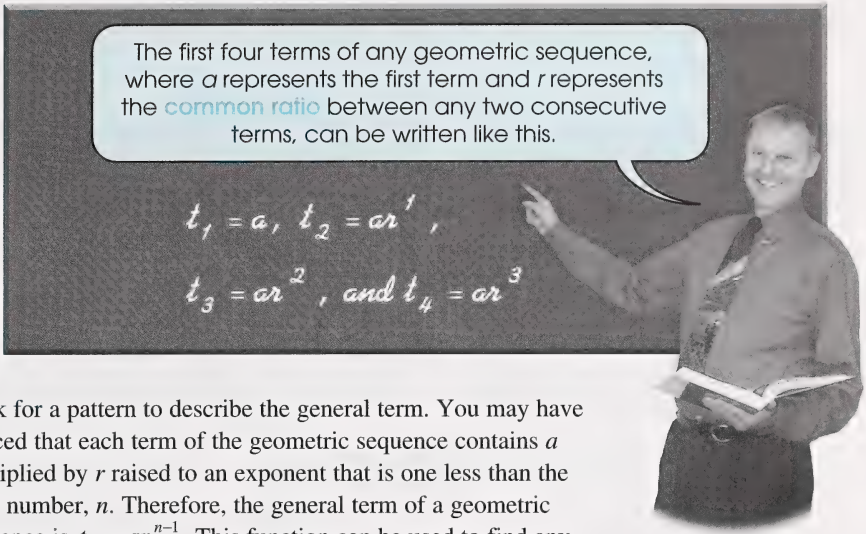
Turn to page 297 of *MATHPOWER*™*12* and read "Geometric Sequences."

1. Answer the following questions on page 297 of the textbook:

a. "Explore: Use Modelling" **b.** questions 1 to 4 of "Inquire"

Turn to page 81 to compare your responses with those in the Appendix.

a constant that is multiplied to each term to produce a geometric sequence



The first four terms of any geometric sequence, where *a* represents the first term and *r* represents the common ratio between any two consecutive terms, can be written like this.

$$t_1 = a,\ t_2 = ar^1,$$
$$t_3 = ar^2,\ \text{and}\ t_4 = ar^3$$

Look for a pattern to describe the general term. You may have noticed that each term of the geometric sequence contains *a* multiplied by *r* raised to an exponent that is one less than the term number, *n*. Therefore, the general term of a geometric sequence is $t_n = ar^{n-1}$. This function can be used to find any term of a geometric sequence given the first term and the common ratio.

Mathematical Process
- ■ Communication
- ■ Connection
- ■ Estimation
- ■ Mental Math
- ■ Problem Solving
- ■ Reasoning
- ☒ Technology
- ■ Visualization

Example

The first two terms of two geometric sequences are as follows:

Sequence 1: $t_1 = 64$ and $t_2 = 32$ Sequence 2: $t_1 = \frac{1}{8}$ and $t_2 = \frac{1}{4}$

a. Use your graphing calculator to graph each sequence on the same set of axes. Compare the graphs of both sequences.

b. How many times greater is the tenth term in Sequence 2 than the tenth term in Sequence 1?

Solution

Before graphing both sequences, make sure your calculator is in Seq mode and Dot mode.

a. Determine the general term of each sequence.

Sequence 1

$$a = t_1 = 64 \text{ and } r = \frac{t_2}{t_1}$$
$$= \frac{32}{64}$$
$$= \frac{1}{2}$$

$$\therefore\ t_n = ar^{n-1}$$
$$= 64\left(\frac{1}{2}\right)^{n-1}$$

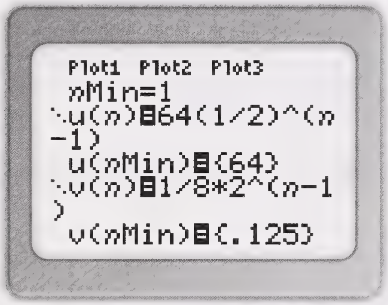
Sequence 2

$$a = t_1 = \frac{1}{8} \text{ and } r = \frac{t_2}{t_1}$$
$$= \frac{\frac{1}{4}}{\frac{1}{8}}$$
$$= 2$$

$$\therefore\ t_n = ar^{n-1}$$
$$= \frac{1}{8}(2)^{n-1}$$

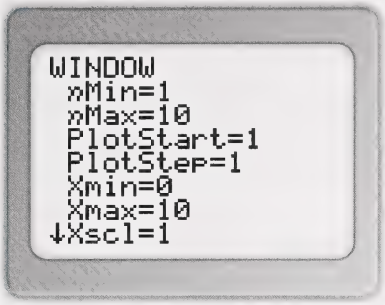
Enter the equations.

Y= 6 4 (1 ÷ 2) ^ (X,T,Θ,*n* − 1) ENTER 6 4 ENTER 1 ÷ 8 × 2 ^ (X,T,Θ,*n* − 1) ENTER 1 ÷ 8 ENTER

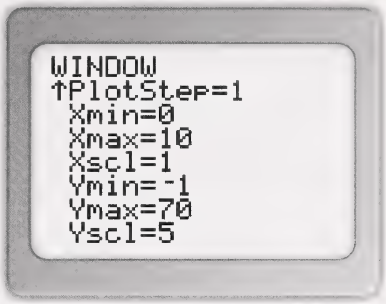


```
Plot1 Plot2 Plot3
nMin=1
\u(n)=64(1/2)^(n
-1)
u(nMin)={64}
\v(n)=1/8*2^(n-1
)
v(nMin)={.125}
```

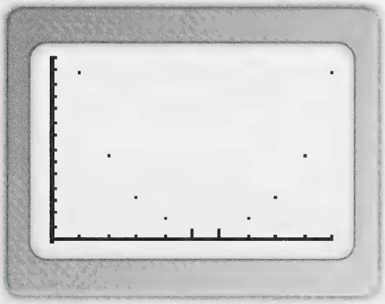
Press GRAPH to display the graphs. **Note:** You may need to adjust the window settings.



```
WINDOW
 nMin=1
 nMax=10
 PlotStart=1
 PlotStep=1
 Xmin=0
 Xmax=10
↓Xscl=1
```

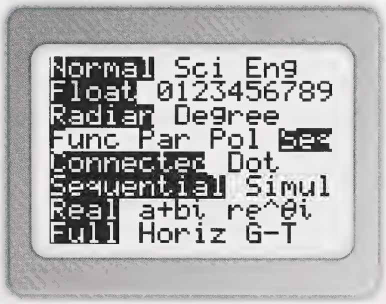


```
WINDOW
↑PlotStep=1
 Xmin=0
 Xmax=10
 Xscl=1
 Ymin=⁻1
 Ymax=70
 Yscl=5
```

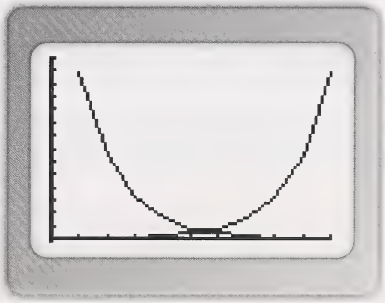


By looking at the graph, you can see that the points appear to lie on exponential curves. The graph of Sequence 1 decreases, whereas the graph of Sequence 2 increases. The points lie on the functions $y=64\left(\frac{1}{2}\right)^{x-1}$ and $y=\frac{1}{8}(2)^{x-1}$.

To display these curves, change your graphing calculator to Connected mode and press GRAPH.



```
Normal Sci Eng
Float 0123456789
Radian Degree
Func Par Pol Seq
Connected Dot
Sequential Simul
Real a+bi re^θi
Full Horiz G-T
```



b. Method 1

Find the tenth term in each sequence.

Sequence 1

$$\begin{aligned} t_n &= 64\left(\frac{1}{2}\right)^{n-1} \\ t_{10} &= 64\left(\frac{1}{2}\right)^{10-1} \\ &= 64\left(\frac{1}{2}\right)^{9} \\ &= 0.125 \end{aligned}$$

Sequence 2

$$\begin{aligned} t_n &= \frac{1}{8}(2)^{n-1} \\ t_{10} &= \frac{1}{8}(2)^{10-1} \\ &= \frac{1}{8}(2)^{9} \\ &= 64 \end{aligned}$$

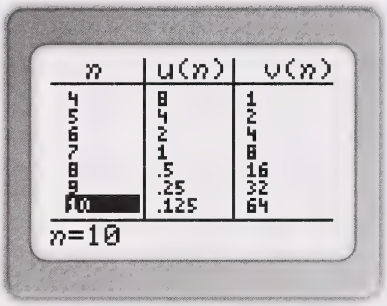
$$\begin{aligned} \therefore \frac{t_{10} \text{ of Sequence 2}}{t_{10} \text{ of Sequence 1}} &= \frac{64}{0.125} \\ &= 512 \end{aligned}$$

The tenth term in Sequence 2 is 512 times greater than the tenth term in Sequence 1.

Method 2

You can obtain the tenth term of each sequence by using the table of values on your graphing calculator.

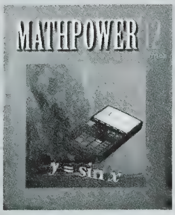
Press 2nd [TABLE].



```
n     u(n)   v(n)
4     8      1
5     4      2
6     2      4
7     1      8
8     .5     16
9     .25    32
10    .125   64
n=10
```

$$\begin{aligned} \therefore \frac{t_{10} \text{ of Sequence 2}}{t_{10} \text{ of Sequence 1}} &= \frac{64}{0.125} \\ &= 512 \end{aligned}$$

The tenth term in Sequence 2 is 512 times greater than the tenth term in Sequence 1.



Turn to pages 298 and 299 of *MATHPOWER™ 12* and work through Examples 1 to 4.

2. Answer the following questions on pages 300 and 301 of the textbook:

a. questions 1, 4, 10, 15, 17, and 22 of "Practice"
b. questions 23, 24, 27, 29, 31, 33, and 35 of "Applications and Problem Solving"

Turn to page 82 to compare your responses with those in the Appendix.

Now Try This

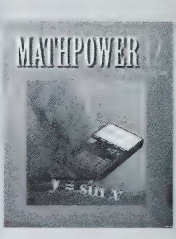
Mathematical Process

■ Communication
■ Connection
■ Estimation
■ Mental Math
☒ Problem Solving
☒ Reasoning
■ Technology
■ Visualization

The ability to recognize a pattern in a sequence that is neither arithmetic nor geometric requires practice using a logical trial-and-error process. The ability to recognize patterns is an important skill; it is even stressed in elementary school. For example, pattern recognition helps children learn to read and spell.

Try your pattern-recognition skills and logical reasoning in the following problem.

3. Turn to page 305 of *MATHPOWER™ 12* and answer "LOGIC POWER." Briefly describe the logic you used to solve this problem.

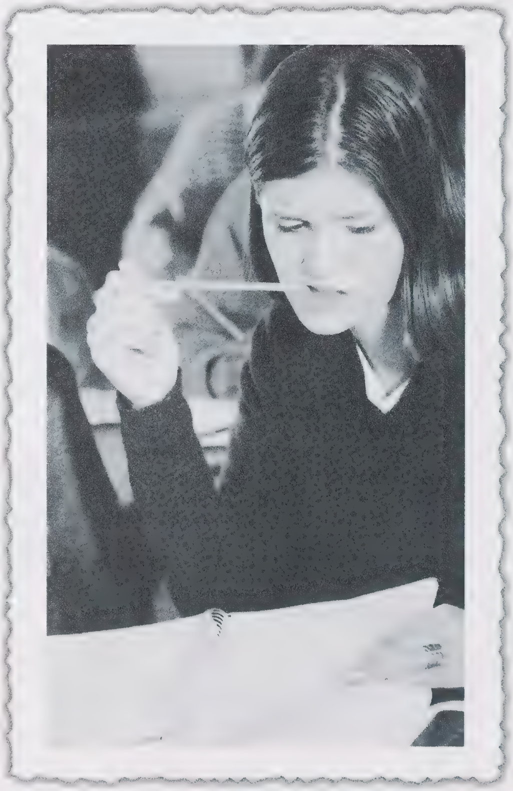


Turn to page 89 to compare your response with the one in the Appendix.

Looking Back

In this activity, you explored geometric sequences. You derived and applied the general term formula, $t_n = ar^{n-1}$, and discovered that the graphs of geometric sequences with positive common ratios other than 1 lie along an exponential curve. This relationship led you to investigate, in detail, the relationship between geometric sequences and exponential functions.

In your journal, explain whether or not the points of a geometric sequence, (n, t_n), where the common ratio is less than zero, lie on an exponential curve. Try graphing $t_n = 3(-2)^{n-1}$, where $1 \le n \le 10$ and $n \in N$. Does the graph of $y = t_n$ support your conclusion?



Mathematical Process

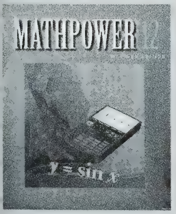
- ☒ Communication
- ☐ Connection
- ☐ Estimation
- ☐ Mental Math
- ☐ Problem Solving
- ☐ Reasoning
- ☐ Technology
- ☐ Visualization

Activity 3: Geometric Sequences and Compound Interest

Has a parent or teacher ever informed you that it is never too early to begin saving towards a secure retirement? As a matter of fact, the younger you are, the wiser it is to invest in your future. Time is the critical factor! If your investments are earning compound interest, even a small amount will grow to a respectable sum over time.

In this activity, you will explore compound interest as an example of a geometric sequence.

Mathematical Process
- ■ Communication
- ☒ Connection
- ■ Estimation
- ■ Mental Math
- ■ Problem Solving
- ■ Reasoning
- ■ Technology
- ■ Visualization



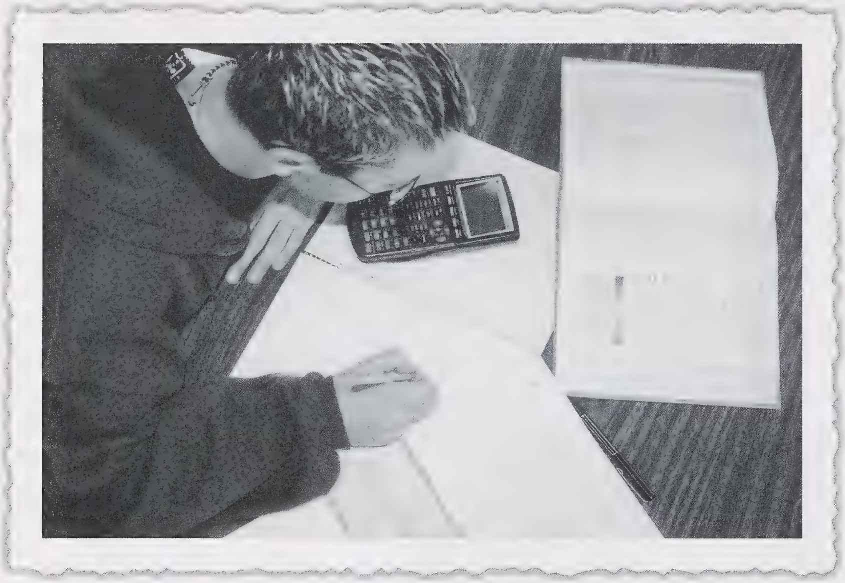
Turn to page 302 of *MATHPOWER™ 12* and read “Geometric Sequences and Compound Interest.”

1. Answer the following questions on page 302 of the textbook:

   a. “Explore: Use a Spreadsheet”
   b. questions 1 to 3 of “Inquire”

Turn to page 92 to compare your answers with those in the Appendix.

In question 1.a., you were asked to set up a spreadsheet to track the growth of an investment, showing the year number, the amount at the beginning of each year, the interest earned each year, and the amount at the end of the year. Did you know that you can use a graphing calculator to do this? However, you will not be able to see the four columns of the table at the same time.

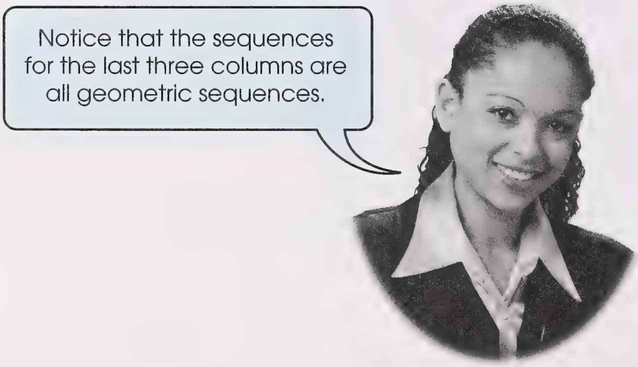


Step 1: Define the functions required for each of the four columns.

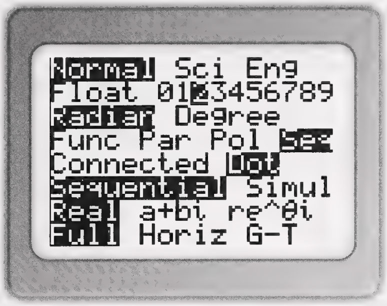
The year number is the set of natural numbers; the opening balance for each year can be represented by the sequence $1000, 1000(1.05), 1000(1.05)^2, \ldots$; the interest earned each year can be represented by the sequence $50, 50(1.05)^1, 50(1.05)^2, \ldots$; and the closing balance for each year can be represented by the sequence $1000(1.05), 1000(1.05)^2, 1000(1.05)^3, \ldots$.

Mathematical Process

- ■ Communication
- ■ Connection
- ■ Estimation
- ■ Mental Math
- ■ Problem Solving
- ■ Reasoning
- ☒ Technology
- ■ Visualization


Notice that the sequences for the last three columns are all geometric sequences.

Step 2: Because you are working with sequences and money, set your graphing calculator to 2 decimal places and in Seq mode.



```
Normal Sci Eng
Float 0123456789
Radian Degree
Func Par Pol Seq
Connected Dot
Sequential Simul
Real a+bi re^θi
Full Horiz G-T
```

Step 3: Determine the general term of each sequence.

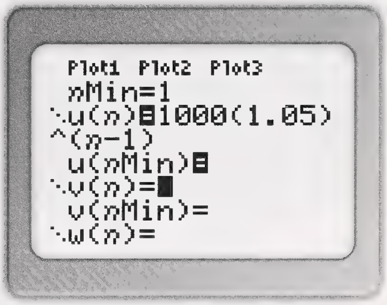
Opening Balance: $t_n = 1000(1.05)^{n-1}$

Interest: $t_n = 50(1.05)^{n-1}$

Closing Balance: $t_n = 1000(1.05)^n$

Step 4: Enter the general term for the Opening Balance column as $u(n)$.

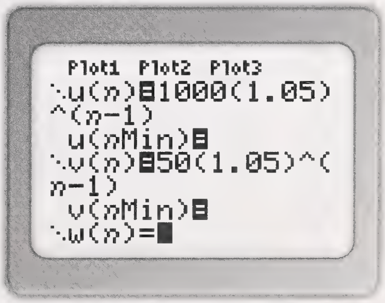
Y= 1 0 0 0 (
1 • 0 5) ^ (
X,T,Θ,n − 1) ENTER
ENTER



```
Plot1 Plot2 Plot3
nMin=1
·.u(n)=1000(1.05)
^(n-1)
 u(nMin)=
·.v(n)=█
 v(nMin)=
·.w(n)=
```

Step 5: Enter the general term for the Interest column as $v(n)$.

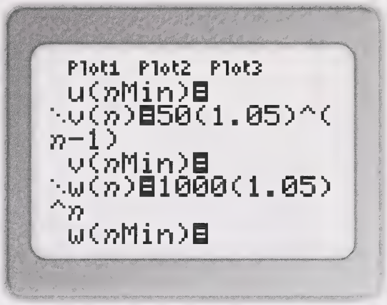
5 0 (1 • 0 5
) ^ (X,T,Θ,n − 1
) ENTER ENTER



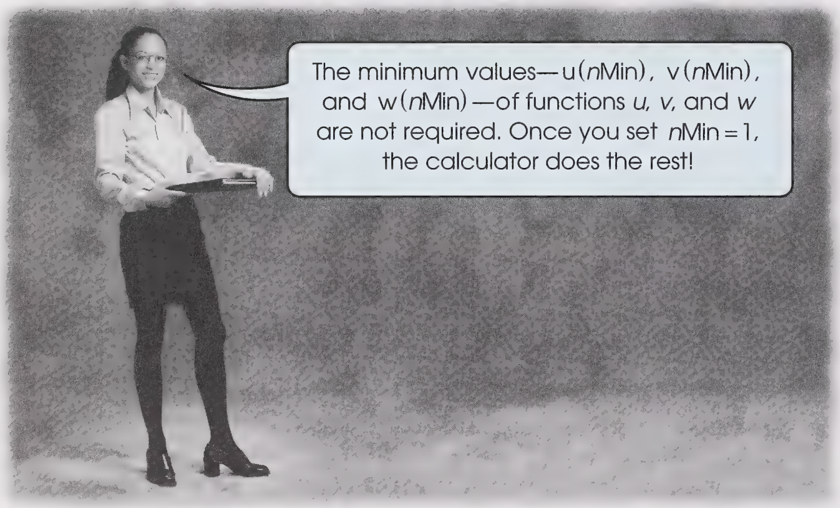
```
Plot1 Plot2 Plot3
·.u(n)=1000(1.05)
^(n-1)
 u(nMin)=
·.v(n)=50(1.05)^(
n-1)
 v(nMin)=
·.w(n)=█
```

Step 6: Enter the general term for the Closing Balance column as $w(n)$.

1 0 0 0 (1 • 0 5) ^ X,T,Θ,*n* ENTER

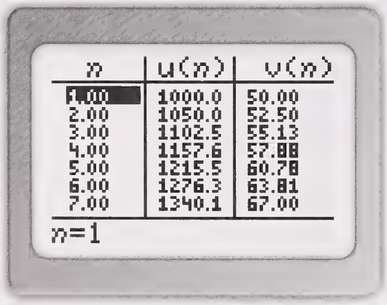


```
 Plot1 Plot2 Plot3
 u(nMin)=
\.v(n)=50(1.05)^(
n−1)
 v(nMin)=
\.w(n)=1000(1.05)
^n
 w(nMin)=
```

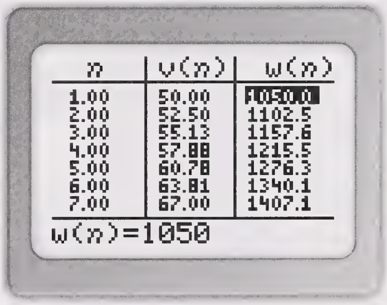


The minimum values—u(*n*Min), v(*n*Min), and w(*n*Min)—of functions *u*, *v*, and *w* are not required. Once you set *n*Min = 1, the calculator does the rest!

Step 7: Press 2nd [TABLE] to obtain the table. **Note:** Use the arrow keys to scroll to the third commn, $w(n)$.



```
  n    | u(n)   | v(n)
 1.00  | 1000.0 | 50.00
 2.00  | 1050.0 | 52.50
 3.00  | 1102.5 | 55.13
 4.00  | 1157.6 | 57.88
 5.00  | 1215.5 | 60.78
 6.00  | 1276.3 | 63.81
 7.00  | 1340.1 | 67.00
n=1
```



```
  n    | v(n)  | w(n)
 1.00  | 50.00 | 1050.0
 2.00  | 52.50 | 1102.5
 3.00  | 55.13 | 1157.6
 4.00  | 57.88 | 1215.5
 5.00  | 60.78 | 1276.3
 6.00  | 63.81 | 1340.1
 7.00  | 67.00 | 1407.1
w(n)=1050
```

You have just worked through a problem where the interest was compounded once a year. Now, you will work through examples and questions where interest is quoted as an annual rate but is not compounded annually. The compounding periods may be weekly, biweekly, monthly, and so on.

Example

Gina has $3000 to invest. If she is quoted an interest rate of 10%/a, compounded quarterly, calculate the amount of Gina's investment after 2 years.

Solution

Calculate the interest rate per period.

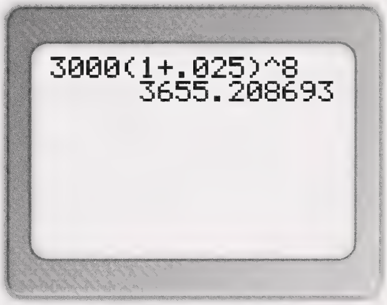
$$\begin{aligned} i &= \frac{10\%}{4} \\ &= 2.25\% \text{ or } 0.025 \end{aligned}$$

Calculate the number of compounding periods.

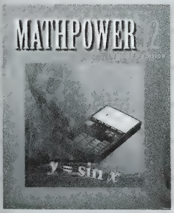
$$\begin{aligned} n &= 4 \times 2 \\ &= 8 \end{aligned}$$

$$\begin{aligned} \therefore\ A &= P(1+i)^n \\ &= 3000(1+0.025)^8 \\ &= 3655.21 \end{aligned}$$

After 2 years, the amount of Gina's investment is $3655.21.



```
3000(1+.025)^8
        3655.208693
```



Turn to pages 302 to 304 of *MATHPOWER™ 12* and work through Examples 1 and 2.

2. Answer the following questions on pages 304 and 305 of the textbook:

- **a.** questions 1, 3, 5, 7, 9, 11, 15, 17, and 19 of "Practice"
- **b.** questions 27 to 31 and 33 of "Applications and Problem Solving"

Turn to page 93 to compare your responses with those in the Appendix.

Now Try This

Mathematical Process

- ■ Communication
- ☒ Connection
- ☒ Estimation
- ☒ Mental Math
- ■ Problem Solving
- ■ Reasoning
- ■ Technology
- ■ Visualization

Many people in the financial community can quickly estimate the time required for an investment earning compound interest to double in value by applying the "rule of 72." They simply divide 72 by the annual interest rate. For example, an annual rate of 9% will double your money in about $72 \div 9$ or 8 years. At 6%, your money will double in about $72 \div 6$ or 12 years.

To find out how accurate this rule is, complete the following question.

3. Turn to page 305 of *MATHPOWER*™ *12* and answer question 35 of "Applications and Problem Solving"

Turn to page 97 to compare your responses with those in the Appendix.

Looking Back

Mathematical Process

- ☒ Communication
- ■ Connection
- ■ Estimation
- ■ Mental Math
- ■ Problem Solving
- ■ Reasoning
- ■ Technology
- ■ Visualization

In this activity, you explored the relationship between compound interest and geometric sequences. You discovered that compound interest is another example of the relationship between exponential and geometric growth.

In your journal, briefly explain why the growth of an investment is sometimes referred to as geometric growth and other times as exponential growth.

Follow-up Activities

If you had difficulties understanding the concepts and skills in the activities, it is recommended that you do the Extra Help. If you have a clear understanding of the concepts and skills, it is recommended that you do the Enrichment. You may decide to do both.

Extra Help

In this section, you explored arithmetic sequences, geometric sequences, and sequences that were neither arithmetic nor geometric. You then discovered that the general term of a sequence could be written as an explicit formula of *n* or written recursively (meaning the value of the term is dependent on knowing the value of the previous term).

Example

Determine whether the following general terms describe an arithmetic sequence, geometric sequence, or neither:

a. $t_n = 4n^3$ **b.** $t_n = 5t_{n-1}$, where $t_1 = 3$ **c.** $t_n = 0.5n - 2$

Solution

a. Determine the first three terms of the sequence.

$$\begin{aligned} t_n &= 4n^3 \\ t_1 &= 4(1)^3 \\ &= 4 \end{aligned} \qquad \begin{aligned} t_n &= 4n^3 \\ t_2 &= 4(2)^3 \\ &= 4(8) \\ &= 32 \end{aligned} \qquad \begin{aligned} t_n &= 4n^3 \\ t_3 &= 4(3)^3 \\ &= 4(27) \\ &= 108 \end{aligned}$$

If this sequence is arithmetic, then $t_2 - t_1 = t_3 - t_2$. Check.

LS	RS
$t_2 - t_1$	$t_3 - t_2$
$= 32 - 4$	$= 108 - 32$
$= 28$	$= 76$

LS $\neq$ RS

If this sequence is geometric, then $\frac{t_2}{t_1} = \frac{t_3}{t_2}$. Check.

LS	RS
$\dfrac{t_2}{t_1}$	$\dfrac{t_3}{t_2}$
$= \dfrac{32}{4}$	$= \dfrac{108}{32}$
$= 8$	$= 3.375$

LS $\neq$ RS

This sequence is neither arithmetic nor geometric.

b. Determine the first three terms of the sequence.

$$t_1 = 3$$

$$\begin{aligned} t_n &= 5t_{n-1} \\ t_2 &= 5t_{2-1} \\ &= 5t_1 \\ &= 5(3) \\ &= 15 \end{aligned}$$

$$\begin{aligned} t_n &= 5t_{n-1} \\ t_3 &= 5t_{3-1} \\ &= 5t_2 \\ &= 5(15) \\ &= 75 \end{aligned}$$

If the sequence is arithmetic, then $t_2 - t_1 = t_3 - t_2$. If the sequence is geometric, then $\frac{t_2}{t_1} = \frac{t_3}{t_2}$. Check.

LS	RS
$t_2 - t_1$	$t_3 - t_2$
$= 15 - 3$	$= 75 - 15$
$= 12$	$= 60$

LS $\neq$ RS

LS	RS
$\dfrac{t_2}{t_1}$	$\dfrac{t_3}{t_2}$
$= \dfrac{15}{3}$	$= \dfrac{75}{15}$
$= 5$	$= 5$

LS $=$ RS

This sequence is geometric.

c. $$\begin{aligned} t_n &= 0.5n - 2 \\ t_1 &= 0.5(1) - 2 \\ &= 0.5 - 2 \\ &= -1.5 \end{aligned} \qquad \begin{aligned} t_n &= 0.5n - 2 \\ t_2 &= 0.5(2) - 2 \\ &= 1 - 2 \\ &= -1 \end{aligned} \qquad \begin{aligned} t_n &= 0.5n - 2 \\ t_3 &= 0.5(3) - 2 \\ &= 1.5 - 2 \\ &= -0.5 \end{aligned}$$

LS	RS
$t_2 - t_1$	$t_3 - t_2$
$= -1 - (-1.5)$	$= -0.5 - (-1)$
$= 0.5$	$= 0.5$

LS = RS

LS	RS
$\frac{t_2}{t_1}$	$\frac{t_3}{t_2}$
$= \frac{-1}{-1.5}$	$= \frac{-0.5}{-1}$
$\doteq 0.67$	$= 0.5$

LS ≠ RS

This sequence is arithmetic.

Example

Determine the number of terms in the following geometric sequence:

$$\frac{1}{9}, \frac{1}{3}, \ldots, 129\ 140\ 163$$

Solution

To determine the number of terms in the sequence, you need to find the value of n when $t_n = 129\ 140\ 163$.

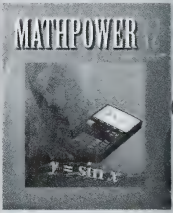
$$a = \frac{1}{9} \text{ and } r = \frac{\frac{1}{3}}{\frac{1}{9}}$$

$$\begin{aligned} &= \frac{1}{\underset{1}{\cancel{3}}} \times \frac{\overset{3}{\cancel{9}}}{1} \\ &= 3 \end{aligned}$$

$$\begin{aligned}
\therefore\ t_n &= ar^{n-1} \\
129\,140\,163 &= \frac{1}{9}(3)^{n-1} \\
3^{17} &= \frac{1}{9}(3)^{n-1} && \leftarrow \text{Write 129 140 163 as a power of 3 using guess and check.} \\
3^{17} \times 9 &= (3)^{n-1} \\
3^{17} \times 3^{2} &= 3^{n-1} \\
3^{19} &= 3^{n-1} \\
19 &= n-1 && \leftarrow \text{[illegible]} \\
20 &= n
\end{aligned}$$

There are 20 terms in this sequence.

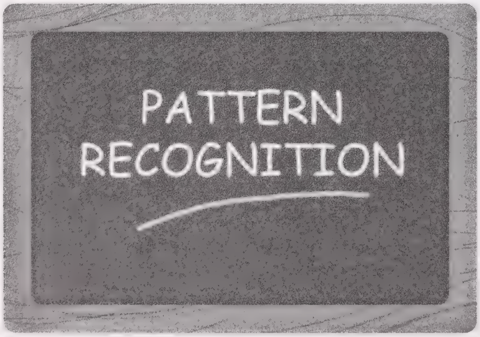
Turn to page 324 of *MATHPOWER™ 12* and answer questions 7, 9.a., 18, 20, 22, 24, 26, and 28 of "Review."



Turn to page 100 to compare your responses with those in the Appendix.

Enrichment

If a sequence is neither arithmetic nor geometric, it is often difficult to determine its general term. Looking for patterns in the differences between successive terms of the sequence is one approach that may work for some sequences.



Example

Determine the general term for the following sequence:

3, 5, 7, 9, 11, …

Solution

n	1	2	3	4	5
t_n	3	5	7	9	11

Δt_n 2 2 2 2

In this case, the difference between successive terms (written Δt_n) is 2. This sequence is arithmetic, and its general term is of the form $t_n = a + (n-1)d$.

In this case,

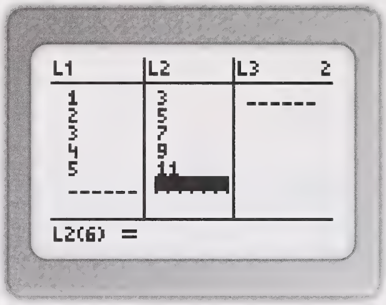
$$\begin{aligned} a = 3 \text{ and } d &= \Delta t_n \\ &= 2 \end{aligned}$$

$$\begin{aligned} \therefore\ t_n &= a + (n-1)d \\ &= 3 + (n-1)2 \\ &= 3 + 2n - 2 \\ &= 2n + 1 \end{aligned}$$

For some of these patterns, you can use your graphing calculator to generate the general term.

How would you use your graphing calculator to find the same result as in the preceding example?

Step 1: Enter the table of values as a list into your calculator. Press STAT 1 (1:Edit); then enter the values of n in column L1 and the values of t_n in the column L2.

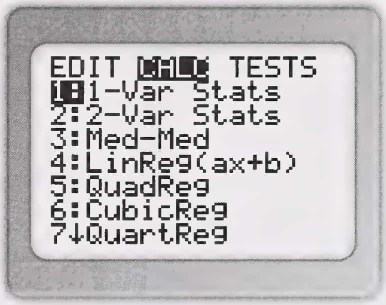


```
L1      L2      L3      2
1       3       ------
2       5
3       7
4       9
5       11
------
L2(6) =
```

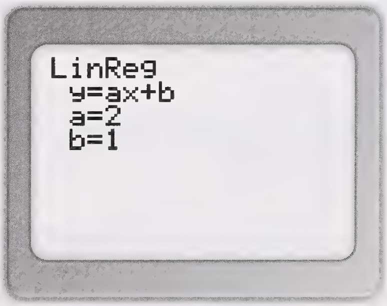
Step 2: Use the features from the CALC menu of the STAT window.

STAT [Select the CALC menu.]

Because the required formula is linear $(t_n = 2n + 1)$, press 4. This is the Linear Regression feature $(\text{LinReg}(\text{ax} + \text{b}))$.



Now, press ENTER to get the equation of the line through the points $(1, 3)$, $(2, 5)$, $(3, 7)$, and so on.



```
LinReg
 y=ax+b
 a=2
 b=1
```

Therefore, the line is $y = 2x + 1$ and the general term of the sequence is $t_n = 2n + 1$.

Mathematical Process

- ☐ Communication
- ☐ Connection
- ☐ Estimation
- ☐ Mental Math
- ☐ Problem Solving
- ☐ Reasoning
- ☒ Technology
- ☐ Visualization

Suppose the sequence is not arithmetic. Can you still use this approach? For example, the general term of the sequence 4, 11, 22, 37, … is $t_n = 2n^2 + n + 1$. Could you have found this general term from the sequence?

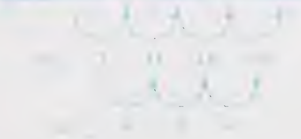
Example

Determine the general term of the following sequence:

4, 11, 22, 37, 56, …

Solution

n	1	2	3	4	5
t_n	4	11	22	37	56

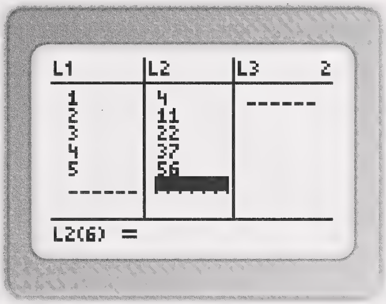


This time the second level of differences, $\Delta^2 t_n$ (the difference between differences), is constant.

Therefore, the general term is not linear; it is quadratic.

First, enter the sequence as a list.

STAT 1 [Enter values of n in L1.] [Enter values of t_n in L2.]

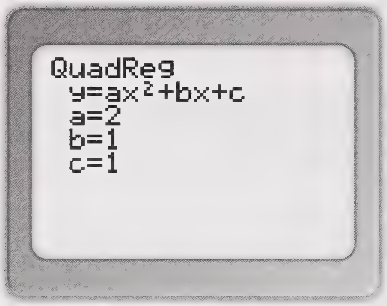


```
L1     L2     L3     2
1      4      ------
2      11
3      22
4      37
5      56
------
L2(6) =
```

Now, determine the quadratic expression for the general term.

STAT [Select the CALC menu.] 5

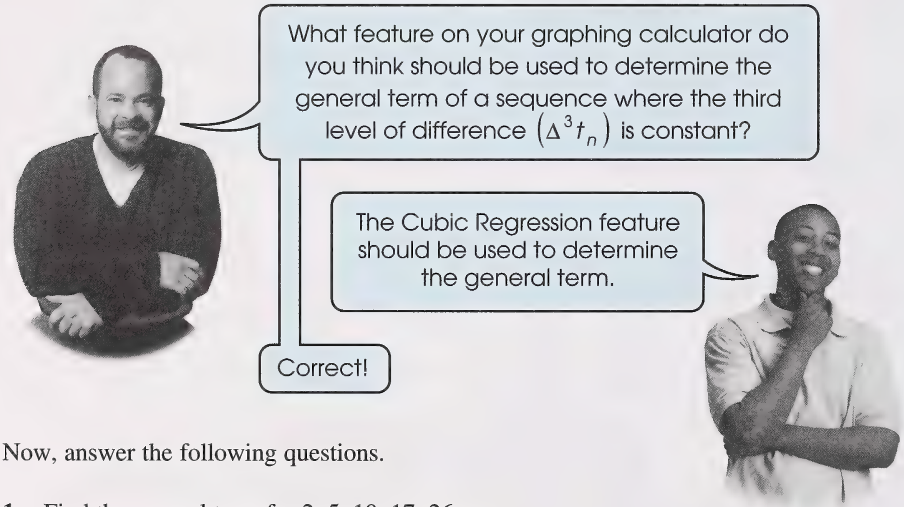
(5:QuadReg) ENTER



```
QuadReg
 y=ax²+bx+c
 a=2
 b=1
 c=1
```

The quadratic function is $y = 2x^2 + x + 1$.
Therefore, the general term is $t_n = 2n^2 + n + 1$.

In the preceding examples, you used the Linear Regression feature when the first level of differences (Δt_n) was constant and the Quadratic Regression feature when the second level of differences $(\Delta^2 t_n)$ was constant. Do you notice a pattern here?


What feature on your graphing calculator do you think should be used to determine the general term of a sequence where the third level of difference $(\Delta^3 t_n)$ is constant?
The Cubic Regression feature should be used to determine the general term.
Correct!

Now, answer the following questions.

1. Find the general term for 2, 5, 10, 17, 26, … .
2. Find the general term for 0, 1, 10, 33, 76, 145.
3. Find the general term for 1, 16, 81, 256, 625, 1296.

Turn to page 104 to compare your responses with those in the Appendix.

Conclusion

In this section, you explored sequences and their application. First, you identified geometric sequences, determined common ratios, and derived and applied formulas for the general term. You then discovered that sequences—geometric sequences in particular—can be determined not only from explicit formulas but also from recursive definitions.

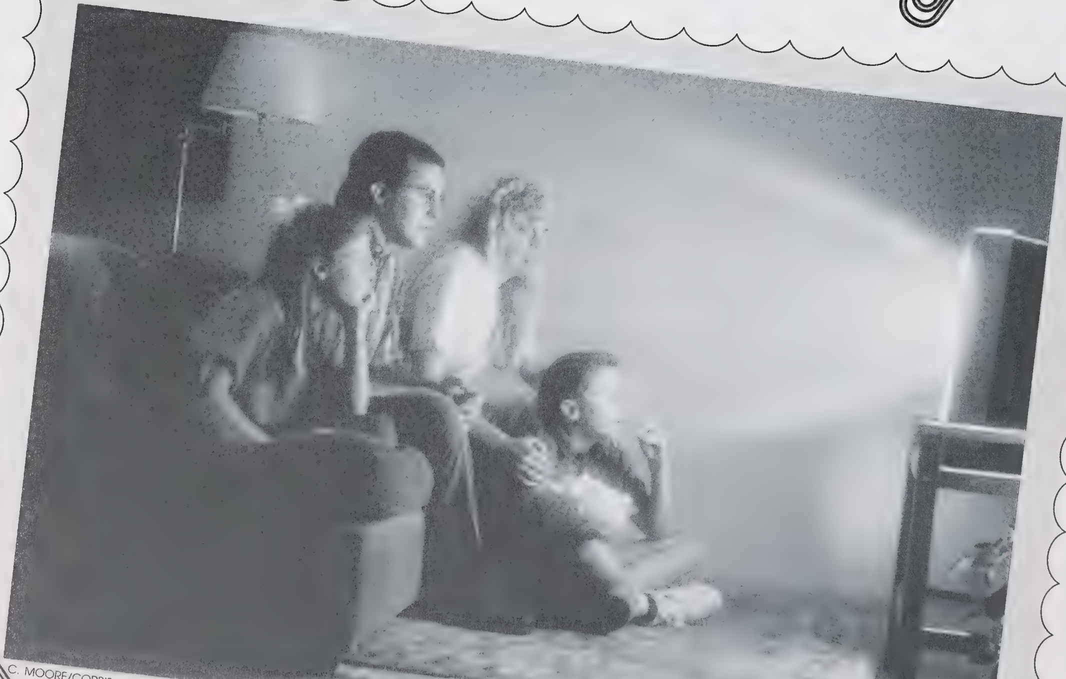
As part of your investigation, you discovered geometric sequences are an example of exponential functions. One of the applications of the exponential function is modelling exponential or geometric growth. Examples of geometric growth are population growth and the growth of investments earning compound interest.

The increase in insect populations, especially when they are first introduced into an environment, are examples of exponential growth. During the period of rapid increase, numbers double, then double again. However, there are limits to growth. Population growth slows, numbers stabilize as predators, disease, weather, and lack of a sustained food supply take their toll.

Assignment

Turn to Assignment Booklet 3A and complete the assignment for Section 1. Remember to submit your completed Assignment Booklet for assessment before starting Section 2.

Section 2



C. MOORE/CORBIS

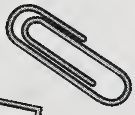
Geometric Series

Many people turn on their televisions each week to watch their favourite television series. Television series differ in many ways. Some television series are comedies, while others are dramas; some consist of half-hour shows, while others consist of one-hour shows; some run weekly, while others run daily. One thing that all television series have in common, though, is they only last as long as there is still interest from the audience. Do you have a favourite television series?

A mini-series is a type of television series that runs daily for a short period of time (usually four days, such as Sunday to Wednesday nights). When viewed individually, the shows form a sequence; but when viewed as a whole, the shows form a series.

Similarly, in mathematics, the sum of a sequence is called a series.

In this section, you will examine geometric series, studying both finite and infinite series. You will then derive expressions that can be used to sum the terms of a finite geometric sequence and to sum the terms of an infinite geometric sequence.



Activity 1: Finite Geometric Series

Mathematical Process
- ☐ Communication
- ☒ Connection
- ☐ Estimation
- ☐ Mental Math
- ☐ Problem Solving
- ☐ Reasoning
- ☐ Technology
- ☐ Visualization

Many people dream of becoming millionaires. Most people believe they would never have to worry about their financial security again if they could only win a big-money contest or lottery or inherit a large sum of money.

Financial advisors, however, will tell you that the road to financial security is much more mundane. People can achieve financial security if they have the self-discipline to set money aside every month for an investment account (such as an RRSP) and allow it to accumulate interest. Even modest deposits will add up to a respectable sum over time.

The regular deposits into an investment account forms a sequence. The terms of this sequence are the individual deposits, numbered from the first deposit to the last.

The sum of the terms of a sequence is called a series. Suppose, instead of being told what the individual deposits in the account were, you were told what the entire amount in the account was. This amount is the value of a series.

series *the sum of the terms of a sequence*

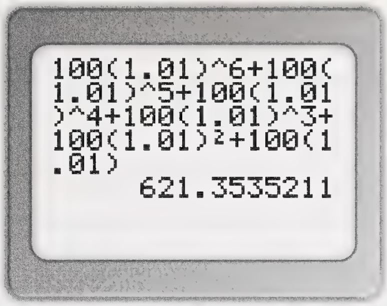
In this activity, you will develop and apply a formula that can be used to determine the sum of the terms of a geometric series.

geometric series *the sum of the terms of a geometric sequence*

One example of a geometric series is the deposits and the interest they earn in a savings account. For example, the value at the end of 6 months where $100 is deposited monthly into a savings account bearing interest at 12%/a, compounded monthly, is as follows:

$$\text{Savings} = 100(1.01)^6 + 100(1.01)^5 + 100(1.01)^4 + 100(1.01)^3 + 100(1.01)^2 + 100(0.01)^1$$

You can determine the value of the expression using your graphing calculator.

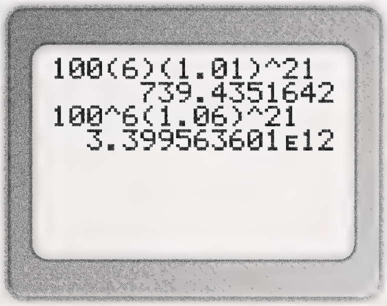


```
100(1.01)^6+100(
1.01)^5+100(1.01
)^4+100(1.01)^3+
100(1.01)²+100(1
.01)
       621.3535211
```

As you can see, the expression will take even longer to calculate if the number of monthly payments increases. Is there an easier way to calculate the sum?

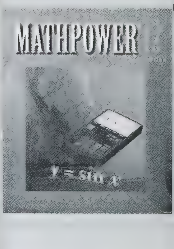
! CAUTION !

Some people try to simplify the expression by rewriting the expression as $100(6)(1.01)^{21}$ or $100^6(1.06)^{21}$. If you calculate the values of these expressions, you will see that neither calculation equals the value obtained when the original expression was calculated.



```
100(6)(1.01)^21
       739.4351642
100^6(1.06)^21
    3.399563601E12
```

These do not equal the original expression because you can only add like terms. Looking back, can you now that $100(1.01)^2$ and $100(1.01)^3$ are not like terms?

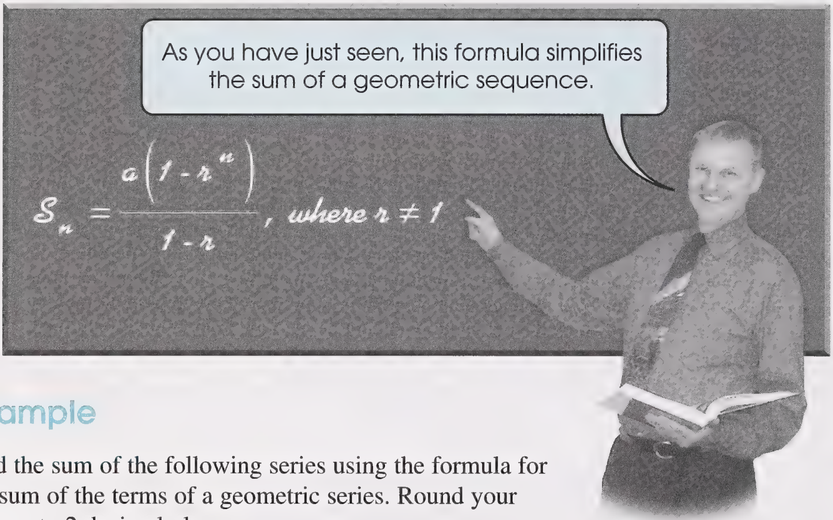


Turn to page 306 of *MATHPOWER™ 12* and read “Geometric Series.”

1. Answer the following questions on page 306 of the textbook:

   a. “Explore: Build a Formula” **b.** questions 1 to 4 of “Inquire”

Turn to page 107 to compare your responses with those in the Appendix.


As you have just seen, this formula simplifies the sum of a geometric sequence.
$S_n = \frac{a(1-r^n)}{1-r}$, where $r \neq 1$

Example

Find the sum of the following series using the formula for the sum of the terms of a geometric series. Round your answer to 2 decimal places.

$$100(1.01)^6 + 100(1.01)^5 + 100(1.01)^4 + 100(1.01)^3 + 100(1.01)^2 + 100(1.01)^1$$

Solution

$$a = 100(1.01)^6$$

$$r = \frac{100(1.01)^5}{100(1.01)^6} = 1.01^{-1}$$

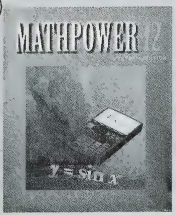
$$n = 6$$

$$\therefore S_n = \frac{a(1-r^n)}{(1-r)}$$

$$S_6 = \frac{100(1.01)^6\left[1-(1.01^{-1})^6\right]}{1-(1.01)^{-1}}$$

$$\doteq 621.35$$

The sum is approximately 621.35.



Turn to page 306 of *MATHPOWER™ 12* and read from the red line to the red line on page 308, working through Examples 1 and 2.

In mathematics, series are often written in a shortened form using what is commonly referred to as sigma notation. For example, you can write the series $S_n = t_1 + t_2 + \ldots + t_n$ as $S_n = \sum_{k=1}^{n} t_k$.

The Greek letter Σ, sigma, corresponds to the letter *S*; both are used to describe a sum. The symbol t_k represents the explicit formula for the general term. The limits $k = 1$ and *n* mean you must replace the variable *k* in the general term by each of the integers from 1 to *n* inclusive.

Example

Determine the value of $\sum_{k=4}^{10} 3^{k-2}$.

Solution

First, expand $\sum_{k=4}^{10} 3^{k-2}$ to determine what type of series it is. The general term is $t_k = 3^{k-2}$ and *k* must be replaced by the integers from 4 through 10 inclusive.

$$\begin{aligned}\sum_{k=4}^{10} 3^{k-2} &= 3^{4-2} + 3^{5-2} + 3^{6-2} + 3^{7-2} + 3^{8-2} + 3^{9-2} + 3^{10-2} \\ &= 3^2 + 3^3 + 3^4 + 3^5 + 3^6 + 3^7 + 3^8\end{aligned}$$

This is a geometric series, where $a = 3^2$, $r = 3$, and $n = 7$.

$$\begin{aligned}\therefore\ S_n &= \frac{a\left(1-r^n\right)}{(1-r)} \\ S_7 &= \frac{3^2\left(1-3^7\right)}{1-3} \\ &= \frac{3^2\left(1-3^7\right)}{-2} \\ &= 9837\end{aligned}$$

Note: The number of terms, *n*, can be determined by counting them or by subtracting the number below the sigma from the number above the sigma and adding 1. Here, $n = 10 - 4 + 1 = 7$.

Example

Kate deposits the same amount of money yearly into an account with an interest rate of 9%/a, compounded annually. If she invests the money for n years, write the total amount of money she will have in n years in sigma notation.

Mathematical Process

- ☐ Communication
- ☒ Connection
- ☐ Estimation
- ☐ Mental Math
- ☒ Problem Solving
- ☒ Reasoning
- ☐ Technology
- ☐ Visualization

Solution

Let P be the amount of money she deposits yearly.

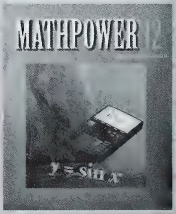
$$\therefore \text{ Total amount} = P(1.09)^n + P(1.09)^{n-1} + \ldots + P(1.09)^1$$

Write the sequence in reverse order.

$$\begin{aligned}\text{Total amount} &= P(1.09)^1 + P(1.09)^2 + \ldots + P(1.09)^n \\ &= \sum_{k=1}^{n} P(1.09)^k\end{aligned}$$

The amount Kate will have is $\sum_{k=1}^{n} P(1.09)^k$.

Turn to page 308 of *MATHPOWER*™ *12* and work through Example 3.



Notice in Example 3, the general term, $8\left(\frac{1}{2}\right)^{k-1}$, for $\sum_{k=1}^{10} 8\left(\frac{1}{2}\right)^{k-1}$ was obtained by using pattern recognition. However, if the series is geometric, you can easily find this term using the formula $t_n = ar^{n-1}$.

For the geometric series $8+4+2+\ldots+\frac{1}{64}$, $a=8$ and $r=\frac{1}{2}$.

$$\therefore\ t_n = 8\left(\frac{1}{2}\right)^{n-1}$$

Next, find the number of terms if $t_n = \frac{1}{64}$.

$$
\begin{aligned}
t_n &= ar^{n-1} \\
\frac{1}{64} &= 8\left(\frac{1}{2}\right)^{n-1} \\
\frac{1}{64 \times 8} &= \left(\frac{1}{2}\right)^{n-1} \\
\frac{1}{512} &= \left(\frac{1}{2}\right)^{n-1} \\
\left(\frac{1}{2}\right)^{9} &= \left(\frac{1}{2}\right)^{n-1} \\
9 &= n-1 \\
n-1 &= 9 \\
n &= 10
\end{aligned}
$$

There are 10 terms in the series.

Therefore, as in Example 3, the series written in sigma notation is as follows:

$$\sum_{k=1}^{\overset{\text{number of terms}}{10}} \underbrace{8\left(\frac{1}{2}\right)^{k-1}}_{\text{[illegible]}}$$

2. Answer the following questions on pages 309 and 310 of the textbook:

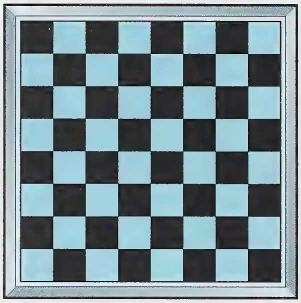
- **a.** questions 1 to 7, 9, 11, 13, 15, 16, and 18 of "Practice"
- **b.** questions 23, 24, 25, 28, 31, and 33 of "Applications and Problem Solving"

3. Two couples plan to invest \$1200 in RRSPs. The Johnstons plan to deposit \$100 into an RRSP at the beginning of each month. The Hays, however, plan to wait until the end of the twelfth month to deposit \$1200 into an RRSP. If the Johnstons receive 6%/a, compounded monthly, how much more money will they have invested than the Hays after 12 months?

Turn to page 108 to compare your responses with those in the Appendix.

Now Try This

In this activity, you derived the formula $S_n = \frac{a(1-r^n)}{1-r}$, where $r \neq 1$, to help you evaluate geometric series. Not all sums are as easy to find however! Check out the next question.



4. How many squares are there on a checkerboard?
   Hint: You must determine the number of 1×1 squares, 2×2 squares, 3×3 squares, and so on; then add them up.

Mathematical Process

■ Communication
☒ Connection
■ Estimation
■ Mental Math
☒ Problem Solving
■ Reasoning
■ Technology
■ Visualization

Turn to page 119 to compare your responses with those in the Appendix.

Looking Back

In this activity, you summed the terms of finite geometric sequences. You then used sigma notation to represent the sum.

Imagine that you have saved enough money for a down payment on a house. In your journal, describe the advantages and disadvantages of having weekly mortgage payments versus monthly payments.

Mathematical Process

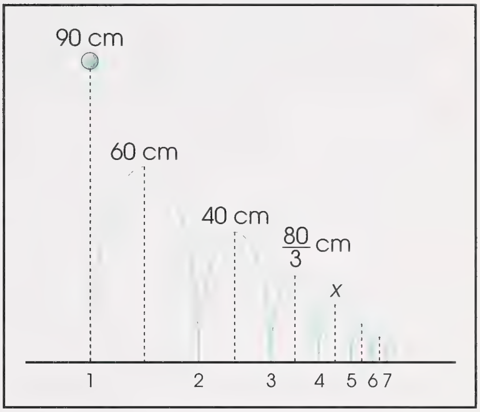
☒ Communication
■ Connection
■ Estimation
■ Mental Math
■ Problem Solving
■ Reasoning
■ Technology
■ Visualization

Activity 2: Infinite Geometric Series

Mathematical Process

- ■ Communication
- ☒ Connection
- ■ Estimation
- ■ Mental Math
- ■ Problem Solving
- ■ Reasoning
- ■ Technology
- ■ Visualization

Have you ever dropped a golf ball on a smooth concrete floor and watched it bounce repeatedly until it came to rest? Suppose you drop a golf ball from a height of 90 cm, and it rebounds two-thirds of its preceding height in each bounce.

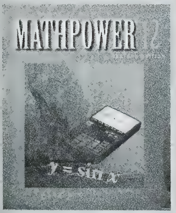


The heights shown in the diagram (90 cm, 60 cm, 40 cm, …) form a geometric sequence. You should be able to determine the total distance through which it travels, say after six bounces, by using the formula for a geometric series.

To determine the distance the bouncing ball travels, work through the segment titled *Geometric Series* from the Computer Assisted Instruction folder on the Pure Mathematics 30 Companion CD.

To determine the total distance the ball bounces before it comes to rest, you must use more than 6 terms to obtain an accurate approximation. As a matter of fact, the answer to this question involves finding the sum of the terms of an infinite geometric series.

In this activity, you will study infinite geometric series. You will discover that some infinite geometric series have finite sums that can be quickly evaluated using a simple formula.



Turn to page 313 of *MATHPOWER™ 12* and read “Infinite Geometric Series.”

1. Answer the following questions on page 313 of the textbook:

   a. “Explore: Use a Pattern”
   b. questions 1 to 4 of “Inquire”

Turn to page 120 to compare your responses with those in the Appendix.

In the preceding questions, you developed the following formula for the sum of an infinite number of terms of a geometric sequence:

$$S=\frac{a}{1-r}, \text{ where } -1<r<1$$

A sum of an infinite geometric series exists if the sequence of its partial sums S_1, S_2, S_3, S_4, …, S_n, … approaches a unique finite value. Consider the following example.

Example

A geometric sequence is defined by $t_n=2\left(\frac{1}{4}\right)^{n-1}$, where $n\in N$.

a. Determine the first 10 partial sums, S_n, $1\le n\le 10$, $n\in N$. Round each partial sum to 4 decimal places.

b. Estimate the sum if *n* is infinite; then determine the sum.

c. What is happening to the terms of the geometric sequence as *n* increases? What is the smallest value of *n*, such that $t_n \doteq 0$ when rounded to 4 decimal places?

Mathematical Process

■ Communication
■ Connection
☒ Estimation
■ Mental Math
■ Problem Solving
☒ Reasoning
■ Technology
■ Visualization

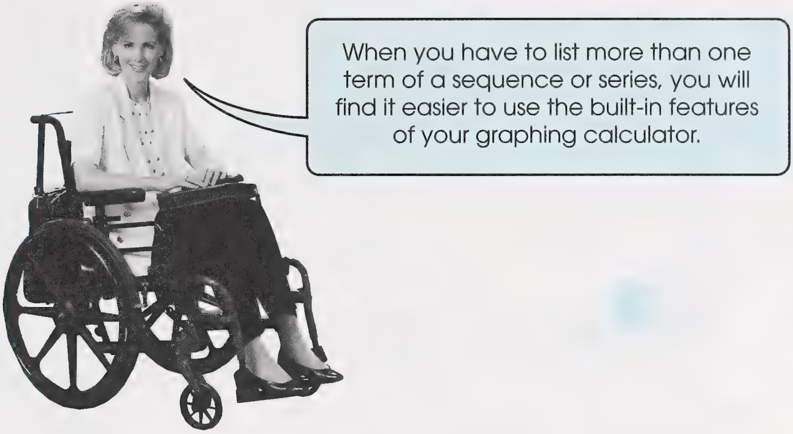
Solution

a. Determine the expression for S_n. Because $t_n=2\left(\frac{1}{4}\right)^{n-1}$, $a=2$ and $r=\frac{1}{4}$.

$$\begin{aligned}\therefore S_n &= \frac{a\left(1-r^n\right)}{1-r}\\ &= \frac{2\left[1-\left(\frac{1}{4}\right)^n\right]}{1-\frac{1}{4}}\\ &= \frac{2\left[1-\left(\frac{1}{4}\right)^n\right]}{\frac{3}{4}}\end{aligned}$$

Mathematical Process

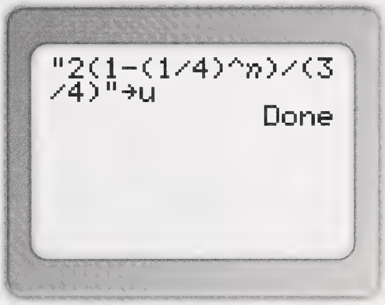
- ■ Communication
- ■ Connection
- ■ Estimation
- ■ Mental Math
- ■ Problem Solving
- ■ Reasoning
- ☒ Technology
- ■ Visualization


When you have to list more than one term of a sequence or series, you will find it easier to use the built-in features of your graphing calculator.

Set your calculator to Seq mode; then press the following keystrokes to list the partial sums, S_n, where $1 \le n \le 10$.

First, enter the expression for S_n.

ALPHA ["] 2 (1 − (1 ÷ 4) ^ X,T,Θ,*n*
) ÷ (3 ÷ 4) ALPHA ["] STO➡ 2nd [u]
ENTER

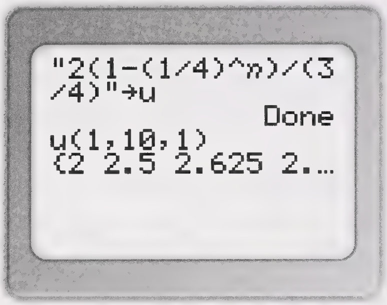


```
"2(1-(1/4)^n)/(3
/4)"→u
                Done
```

Now, enter the values for *n*.

2nd [u] (1 , 1 0 , 1) ENTER

Your display should look like the following.



```
"2(1-(1/4)^n)/(3
/4)"→u
              Done
u(1,10,1)
{2 2.5 2.625 2.…
```

Note: *Press the right arrow key to see the rest of the partial sums.*

The partial sums are given in the followng table.

n	S_n	n	S_n
1	2	6	2.6660
2	2.5	7	2.6665
3	2.625	8	2.6666
4	2.6563	9	2.6667
5	2.6641	10	2.6667

b. If n is infinite, the sum appears to be $2.\overline{6}$ or $2\frac{2}{3}$.

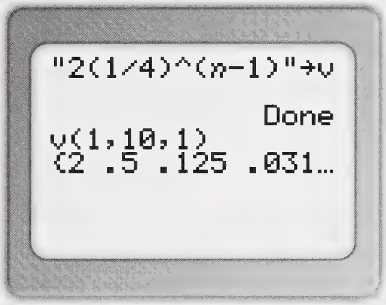
Check

$$\begin{aligned} S &= \frac{a}{1-r} \\ &= \frac{2}{1-\frac{1}{4}} \\ &= \frac{2}{\frac{3}{4}} \\ &= \frac{8}{3} \\ &= 2\frac{2}{3} \end{aligned}$$

c. Using your graphing calculator, enter the expression for t_n and list the first 10 terms. More terms can easily be generated, if necessary.

ALPHA ["] 2 (1 ÷ 4) ^ (X,T,Θ,*n* − 1) ALPHA ["] STO➧ 2nd [v] ENTER

2nd [v] (1 , 1 0 , 1) ENTER



```
"2(1/4)^(n-1)"→v
                Done
v(1,10,1)
{2 .5 .125 .031…
```

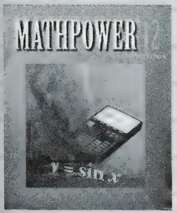
Recall that 4.882 812 5 E^{-4} is the same as 0.000 488 281 25.

As n gets larger, the terms of the geometric sequence, t_n, approach zero.

The smallest value of n, such that $t_n \doteq 0$ when rounded to 4 decimal places, is 9 $\left(t_9 = 0.000\ 030\ 517\ 578\ 13\right)$.

converge: approach a unique finite value

When the value of t_n approaches zero, you know the value of S_n and S_{n-1} will be nearly the same. Both the sequence and series are said to converge.



Turn to page 313 of *MATHPOWER*™ *12* and read from the red line to the bottom of page 315, working through the example.

2. Answer questions 1, 4, 5, 6, 7, 9, 14, and 15 of "Practice" on page 316 of the textbook.

Turn to page 121 to compare your responses with those in the Appendix.

One of the advantages of knowing the formula for the sum of an infinite geometric sequence is that you can quickly change a repeating decimal to its fraction form.

Example

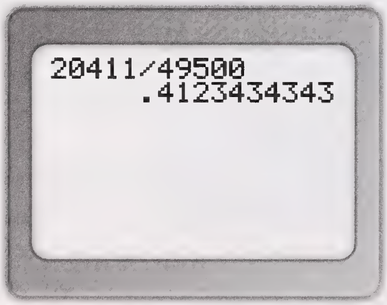
Express the repeating decimal $0.412\,\overline{34}$ as a fraction.

Solution

To express the repeating decimal as a fraction, you must write the decimal as a geometric series first.

$$\begin{aligned}0.412\,\overline{34} &= 0.412 + 0.000\,34 + 0.000\,003\,4 + \ldots \\ &= 0.412 + \text{the sum of an infinite geometric series, where } a = 0.000\,34 \text{ and } r = 0.01 \\ &= 0.412 + \frac{0.000\,34}{1-0.01} \\ &= \frac{412}{1000} + \frac{0.000\,34}{0.99} \\ &= \frac{412(99)}{99\,000} + \frac{34}{99\,000} \\ &= \frac{40\,822}{99\,000} \\ &= \frac{20\,411}{49\,500}\end{aligned}$$

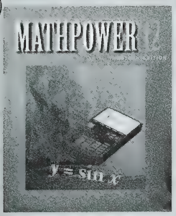
Check using your graphing calculator.



```
20411/49500
        .4123434343
```

Mathematical Process

- ☐ Communication
- ☐ Connection
- ☐ Estimation
- ☐ Mental Math
- ☐ Problem Solving
- ☐ Reasoning
- ☒ Technology
- ☐ Visualization



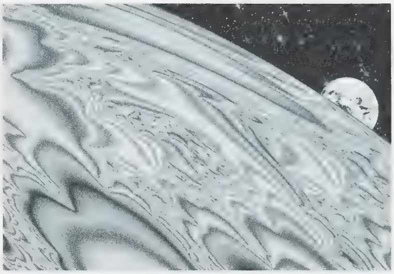
3. Turn to pages 316 and 317 of *MATHPOWER™ 12* and answer 17.c., 17.d., 18, 19, and 22 of "Applications and Problem Solving."

Turn to page 124 to compare your responses with those in the Appendix.

Now Try This

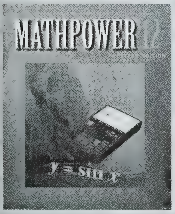
Mathematical Process
- ☐ Communication
- ☒ Connection
- ☐ Estimation
- ☐ Mental Math
- ☐ Problem Solving
- ☐ Reasoning
- ☐ Technology
- ☐ Visualization

Certain interesting patterns can be formed by repeating a set of steps. One type of pattern is called a fractal. Fractals have intricate, self-repeating shapes and have been used in the movie industry to create such things as alien landscapes.


JIM ZUCKERMAN/CORBIS

If you have access to the Internet, you may wish to explore fractals further at the following website:

http://spanky.triumf.ca



4. **a.** Turn to "GETTING STARTED" on page 284 of *MATHPOWER™ 12* and answer questions 1 to 6 of Investigation 2, "Fractals."

b. Turn to page 317 of *MATHPOWER™ 12* and answer question 24 of "Applications and Problem Solving."

Turn to page 131 to compare your responses with those in the Appendix.

Looking Back

Mathematical Process
- ☒ Communication
- ☐ Connection
- ☐ Estimation
- ☐ Mental Math
- ☐ Problem Solving
- ☐ Reasoning
- ☐ Technology
- ☐ Visualization

In this activity, you explored infinite geometric series. You also derived and applied a formula for determining the sum of a convergent infinite geometric series.

In your journal, give examples of infinite geometric series for which the sum is finite and for which the sum is undefined. How are these series similar? How do they differ?

Follow-up Activities

If you had difficulties understanding the concepts and skills in the activities, it is recommended that you do the Extra Help. If you have a clear understanding of the concepts and skills, it is recommended that you do the Enrichment. You may decide to do both.

Extra Help

In this section, you dealt with geometric series, both finite and infinite. In the previous section, you dealt with geometric sequences.

A common problem many students have is knowing when to use the formula for the general term of a sequence, $t_n = ar^{n-1}$, and when to use the formula for the sum of a series, $S_n = \frac{a\left(1-r^n\right)}{1-r}$.

One approach is to look for key words in the problem. Even if the word *sequence* appears in a question, you still may be expected to use the series formula if the word *sum* also appears. Occasionally, you will have to use the general term formula to determine the number of terms in a given series.

Example

The *n*th partial sum of a series is $S_n = -3\left(1-2^n\right)$.

a. Determine the first five partial sums.
b. Write the first five terms of the corresponding sequence.
c. What is the relationship between t_5, S_5, and S_4?

Solution

a.

$$\begin{aligned} S_n &= -3\left(1-2^n\right) \\ S_1 &= -3\left(1-2^1\right) \\ &= -3(1-2) \\ &= -3(-1) \\ &= 3 \end{aligned} \qquad \begin{aligned} S_n &= -3\left(1-2^n\right) \\ S_2 &= -3\left(1-2^2\right) \\ &= -3(1-4) \\ &= -3(-3) \\ &= 9 \end{aligned} \qquad \begin{aligned} S_n &= -3\left(1-2^n\right) \\ S_3 &= -3\left(1-2^3\right) \\ &= -3(1-8) \\ &= -3(-7) \\ &= 21 \end{aligned}$$

$$\begin{aligned} S_n &= -3\left(1-2^n\right) \\ S_4 &= -3\left(1-2^4\right) \\ &= -3(1-16) \\ &= -3(-15) \\ &= 45 \end{aligned} \qquad \begin{aligned} S_n &= -3\left(1-2^n\right) \\ S_5 &= -3\left(1-2^5\right) \\ &= -3(1-32) \\ &= -3(-31) \\ &= 93 \end{aligned}$$

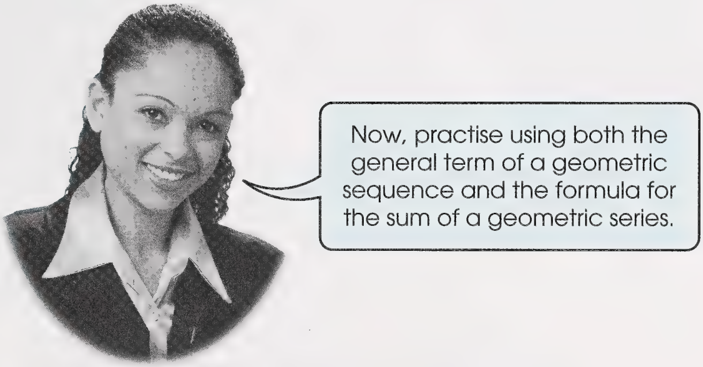
The first five partial sums are 3, 9, 21, 45, and 93.

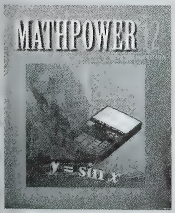
b. $$\begin{aligned} t_1 &= S_1 \\ &= 3 \end{aligned} \qquad \begin{aligned} t_2 &= S_2 - S_1 \\ &= 9-3 \\ &= 6 \end{aligned} \qquad \begin{aligned} t_3 &= S_3 - S_2 \\ &= 21-9 \\ &= 12 \end{aligned} \qquad \begin{aligned} t_4 &= S_4 - S_3 \\ &= 45-21 \\ &= 24 \end{aligned} \qquad \begin{aligned} t_5 &= S_5 - S_4 \\ &= 93-45 \\ &= 48 \end{aligned}$$

The first five terms of the sequence are 3, 6, 12, 24, and 48.

c. The relationship between t_5, S_5, and S_4 is as follows:

$$t_5 = S_5 - S_4 \text{ or } S_5 = S_4 + t_5$$


Now, practise using both the general term of a geometric sequence and the formula for the sum of a geometric series.



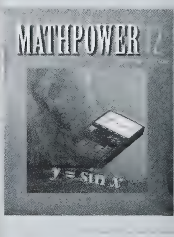
Turn to page 326 of *MATHPOWER™ 12* and answer the questions 16, 17, 18, 20, and 24 of “Chapter Check.”

Turn to page 134 to compare your responses with those in the Appendix.

Enrichment

In Activity 2 of this section, you explored infinite geometric series. You wil now explore infinite series that are not geometric.

Turn to page 318 of *MATHPOWER*™ *12* and read "Exploring Other Infinite Series."

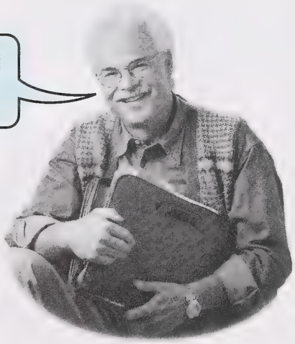


1. Answer the following questions on page 318 of the textbook:

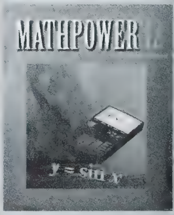
   a. "Explore: Use a Code" **b.** questions 1 to 7 of "Inquire"

Turn to page 138 to compare your responses with those in the Appendix.

Now you see how π has fascinated mathematicians for many years.



Turn to page 319 of *MATHPOWER*™ *12* and read from the top of the page to the red line on page 320, working through Example 1.

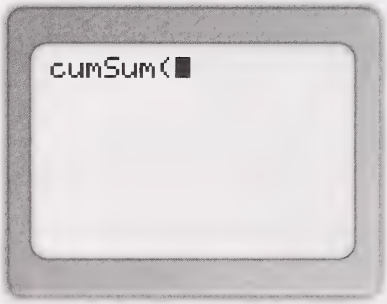


You can make the calculations less tedious by using the Cummulative Sum feature and the Sequence feature on your graphing calculator. The following keystrokes may be helpful.

Mathematical Process

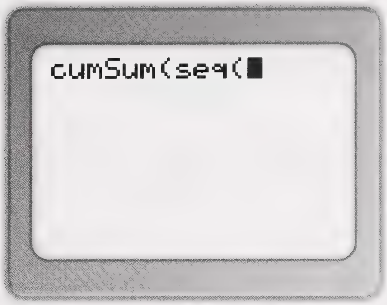
- ■ Communication
- ■ Connection
- ■ Estimation
- ■ Mental Math
- ■ Problem Solving
- ■ Reasoning
- ☒ Technology
- ■ Visualization

First, press 2nd [LIST] [Select the OPS menu.] 6 to select the Cummulative Sum feature.



```
cumSum(
```

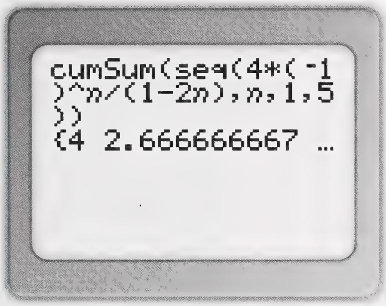
Next, press 2nd [LIST] [Select the OPS menu.] 5 to select the Sequence feature.



```
cumSum(seq(█
```

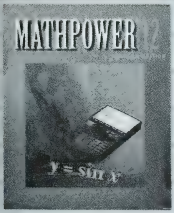
Now, enter the general term of the sequence, the variable, the first value of the variable, and the last value of the variable.

4 × ((–) 1) ^ X,T,Θ,*n* ÷ (1 – 2 X,T,Θ,*n*)
, X,T,Θ,*n* , 1 , 5)) ENTER



```
cumSum(seq(4*(⁻1
)^n/(1-2n),n,1,5
))
{4 2.666666667 …
```

Use the arrow keys to see the other sums.



2. Answer the following questions on pages 320 and 321 of the textbook. Use your graphing calculator wherever possible.

   a. questions 1, 3, 4, and 6 of "Practice"
   b. questions 7, 8, 9.a., 9.b., 9.c., 10, and 12 of "Applications and Problem Solving"

Turn to page 139 to compare your responses with those in the Appendix.

Conclusion

In this section, you began by exploring finite geometric series. You discovered that geometric series are simply the sums of the terms of geometric sequences. To evaluate these series, you derived and applied the formula for the sum of the first n terms of a geometric sequence.

You then discovered that it is possible to evaluate the sum of the terms of an infinite geometric series and that this sum is finite if the common ratio, r, of this sequence is $-1 < r < 1$.

Each time a child swings back and forth on a swing, the arc he or she travels through becomes progressively smaller. Did you know that the lengths of these arcs can be modelled by a geometric sequence? As a matter of fact, the total distance the child travels after swinging back and forth ten times or the total distance the child travels until he or she comes to rest can be approximated by applying the formulas for finite and infinite geometric series respectively.

You see, evaluating finite and infinite geometric series can be child's play!

Assignment

Turn to Assignment Booklet 3B and complete the assignment for Section 2.

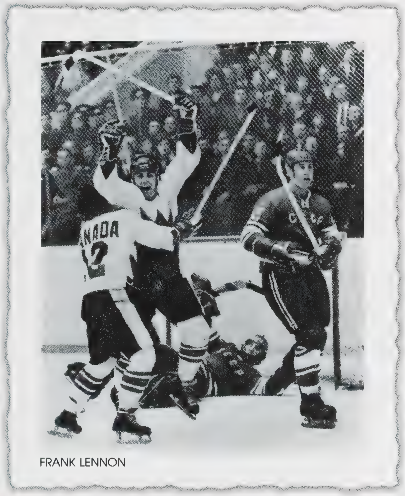
Module Summary

In this module you explored geometric sequences and series.

In Section 1, you began by reviewing sequences, which you studied previously in Pure Mathematics 10. You practised representing sequences using general term (or explicit) and recursive formulas. You then focused on geometric sequences and their application, discovering that geometric sequences are related to the exponential functions you used to model growth and decay in Module 2.

In Section 2, you explored finite and infinite geometric series. Once again, you employed the formulas you derived to solve practical problems.

The words *sequence* and *series* are commonly used both in mathematical and everyday terms. In ordinary conversation, as in mathematics, a sequence often refers to a number of related events that occur in a particular order and a series often refers to the sequence as a whole. Think of how a hockey series consists of a sequence of games that must be played on particular dates in particular locations.



FRANK LENNON

One of the most famous hockey series in Canadian history was the Canada-Soviet hockey series of 1972, also known as the Summit Series. Eight games were played: four in Canada and four in the Soviet Union. Canada and the Soviet Union had identical records from the first seven games, three wins apiece and a tie. Canada won the series in the dying seconds of the last game! This series marked Canada's prominence in international hockey.

Final Module Assignment

Turn to Assignment Booklet 3B and complete the final module assignment. Remember to submit your completed Assignment Booklet for assessment.

Appendix

Glossary
Suggested Answers
Credits

Glossary

amount: the total value of an investment, including the principal and the interest

arithmetic sequence: a sequence for which the difference between successive terms is constant

arithmetic series: the sum of the terms of an arithmetic sequence

common difference: a constant that is added to each term to produce an arithmetic sequence

common ratio: a constant that is multiplied to each term to produce a geometric sequence

converge: approach a unique finite value

convergent sequence: a sequence in which the difference between two consecutive terms is equal to zero when n is large

convergent series: a series in which the sum is finite

divergent series: a series that is not convergent

explicit formula: a formula for the nth term of a sequence, written as an expression of n

finite sequence: a sequence that has a last term

general term: a function that describes all terms in a sequence

geometric means: the terms between any two terms of a geometric sequence

geometric sequence: a sequence in which the ratio of successive terms is constant

geometric series: the sum of the terms of a geometric sequence

infinite sequence: a sequence that has no last term

e.g., $1, 2, 3, 4, \ldots$

infinite series: a series that has no last term

e.g., $1+2+3+4+\ldots$

partial sums: the sum of one or more consecutive terms of a sequence

per annum: per year, denoted by "/a"

principal: the value of the money invested or borrowed

recursive formula: a formula used to define terms of a sequence by relating them to previous terms of the sequence

sequence: a set or list of numbers arranged in a definite order

A sequence is a function whose domain is a subset of the natural numbers, N, and whose range is a subset of the real numbers, R. The sequence itself shows the range of the function.

series: the sum of the terms of a sequence

sigma: a Greek letter, Σ, used as a summation symbol

term: a number in a sequence

The first term may be denoted as "a" or "t_1."

Suggested Answers

Section 1: Activity 1

1. a. **Textbook question "Explore: Use a Pattern," p. 286**

Number of Logs Across the Base	Total Number of Logs in the Pile
1	1
2	3
3	$3+3=6$
4	$6+4=10$
5	$10+5=15$
6	$15+6=21$
7	$21+7=28$

b. **Textbook questions 1 to 3 of "Inquire," p. 286**

1. Yes, the numbers in the second column of the table follow a pattern. The total number of logs in a pile with n layers is found by adding n to the previous total.

2. 8 logs across the base: Total $= 28+8=36$
9 logs across the base: Total $= 36+9=45$
10 logs across the base: Total $= 45+10=55$

There are 55 logs in a pile having 10 logs across the base.

3. $t_n = t_{n-1} + n$

Section 1: Activity 1 (continued)

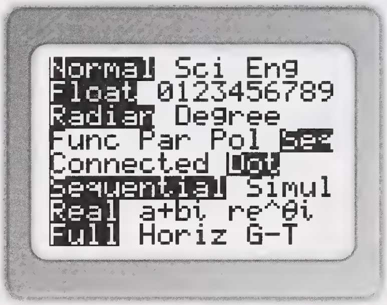
2. **a.** **Textbook questions 3, 13, 14, 16, 19, 20, and 21 of "Practice," p. 290**

3. **a.** $t_1 = 5$

$$\begin{aligned} t_n &= 3t_{n-1} + n \\ t_2 &= 3t_{2-1} + 2 \\ &= 3t_1 + 2 \\ &= 3(5) + 2 \\ &= 17 \end{aligned} \qquad \begin{aligned} t_n &= 3t_{n-1} + n \\ t_3 &= 3t_{3-1} + 3 \\ &= 3t_2 + 3 \\ &= 3(17) + 3 \\ &= 54 \end{aligned} \qquad \begin{aligned} t_n &= 3t_{n-1} + n \\ t_4 &= 3t_{4-1} + 4 \\ &= 3t_3 + 4 \\ &= 3(54) + 4 \\ &= 166 \end{aligned} \qquad \begin{aligned} t_n &= 3t_{n-1} + n \\ t_5 &= 3t_{5-1} + 5 \\ &= 3t_4 + 5 \\ &= 3(166) + 5 \\ &= 503 \end{aligned}$$

The first five terms are 5, 17, 54, 166, and 503.

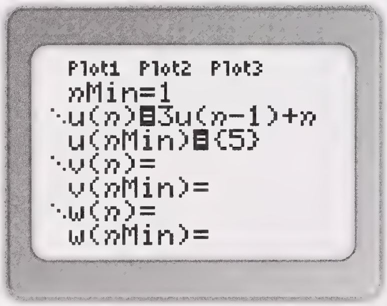
b. Before creating a table of values using your graphing calculator, make sure your calculator is in Seq mode and Dot mode.



```
Normal Sci Eng
Float 0123456789
Radian Degree
Func Par Pol Seq
Connected Dot
Sequential Simul
Real a+bi re^θi
Full Horiz G-T
```

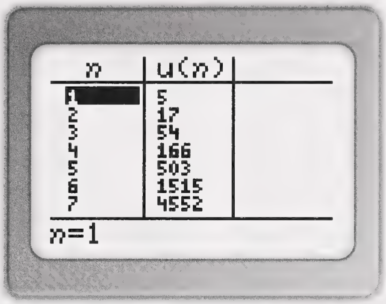
Now, enter the information.

Y= 3 2nd [u] (X,T,Θ,*n* – 1) + X,T,Θ,*n* ENTER
5 ENTER



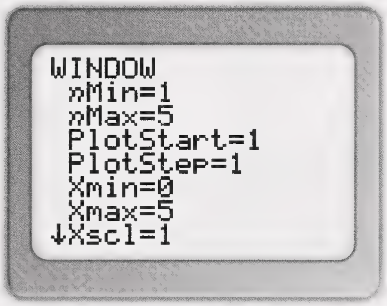
```
 Plot1 Plot2 Plot3
 nMin=1
·.u(n)=3u(n−1)+n
 u(nMin)={5}
·.v(n)=
 v(nMin)=
·.w(n)=
 w(nMin)=
```

Press 2nd [TABLE] to view the table of values.

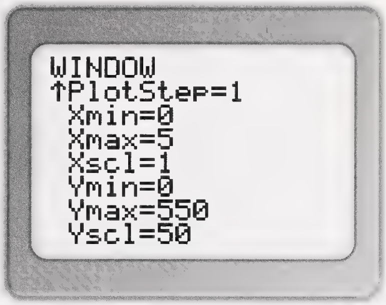


```
  n    | u(n) |
 1     | 5
 2     | 17
 3     | 54
 4     | 166
 5     | 503
 6     | 1515
 7     | 4552
n=1
```

To plot the sequence, use the following window settings.

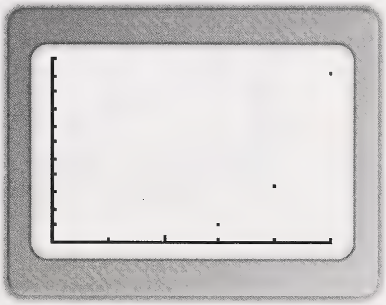


```
WINDOW
 nMin=1
 nMax=5
 PlotStart=1
 PlotStep=1
 Xmin=0
 Xmax=5
↓Xscl=1
```



```
WINDOW
↑PlotStep=1
 Xmin=0
 Xmax=5
 Xscl=1
 Ymin=0
 Ymax=550
 Yscl=50
```

Press GRAPH to see the graph of the sequence.



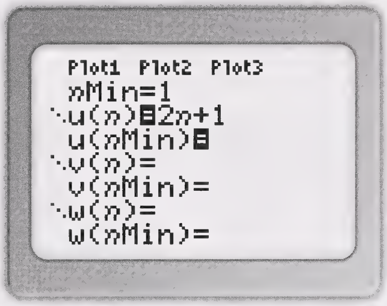
Section 1: Activity 1 (continued)

13. a.

$$\begin{array}{lllll}
t_n = 2n+1 & t_n = 2n+1 & t_n = 2n+1 & t_n = 2n+1 & t_n = 2n+1 \\
t_1 = 2(1)+1 & t_2 = 2(2)+1 & t_3 = 2(3)+1 & t_4 = 2(4)+1 & t_5 = 2(5)+1 \\
\quad = 3 & \quad = 5 & \quad = 7 & \quad = 9 & \quad = 11
\end{array}$$

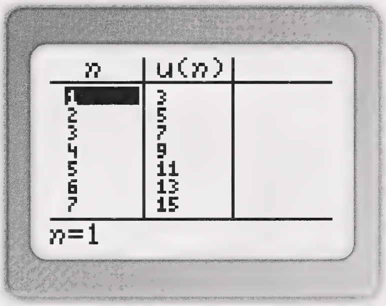
The first five terms are 3, 5, 7, 9, and 11.

b. Press Y= 2 X,T,Θ,*n* + 1 to enter the information.



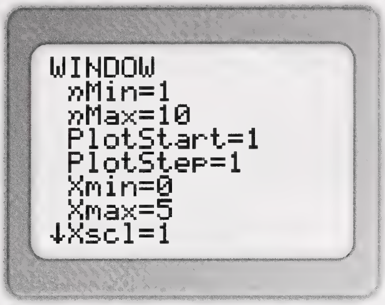
```
 Plot1 Plot2 Plot3
 nMin=1
:.u(n)=2n+1
 u(nMin)=
:.v(n)=
 v(nMin)=
:.w(n)=
 w(nMin)=
```

Press 2nd [TABLE] to view the table of values.

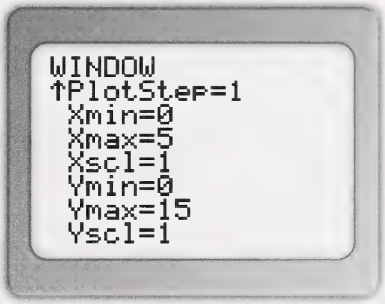


```
   n    | u(n) |
 1      | 3    |
 2      | 5    |
 3      | 7    |
 4      | 9    |
 5      | 11   |
 6      | 13   |
 7      | 15   |
n=1
```

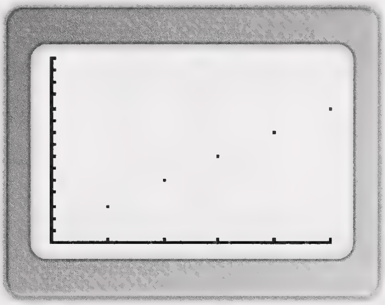
Press GRAPH to see the graph of the sequence. **Note:** You will need to adjust the window settings.



```
WINDOW
 nMin=1
 nMax=10
 PlotStart=1
 PlotStep=1
 Xmin=0
 Xmax=5
↓Xscl=1
```



```
WINDOW
↑PlotStep=1
 Xmin=0
 Xmax=5
 Xscl=1
 Ymin=0
 Ymax=15
 Yscl=1
```



14. a.

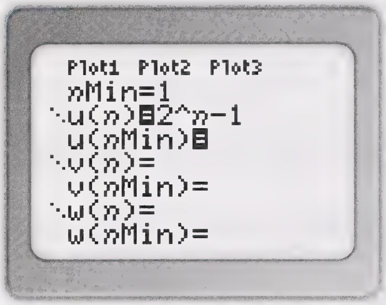
$t_n = 2^n - 1$	$t_n = 2^n - 1$	$t_n = 2^n - 1$	$t_n = 2^n - 1$	$t_n = 2^n - 1$
$t_1 = 2^1 - 1$	$t_2 = 2^2 - 1$	$t_3 = 2^3 - 1$	$t_4 = 2^4 - 1$	$t_5 = 2^5 - 1$
$= 2 - 1$	$= 4 - 1$	$= 8 - 1$	$= 16 - 1$	$= 32 - 1$
$= 1$	$= 3$	$= 7$	$= 15$	$= 31$

The first five terms are 1, 3, 7, 15, and 31.

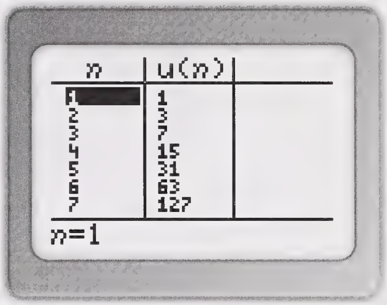
Section 1: Activity 1 (continued)

b. To view the table of values, press the following:

Y= 2 ^ X,T,Θ,*n* − 1 2nd [TABLE]

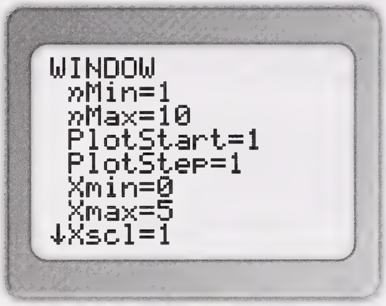


```
Plot1 Plot2 Plot3
 nMin=1
·.u(n)■2^n-1
 u(nMin)■
·.v(n)=
 v(nMin)=
·.w(n)=
 w(nMin)=
```

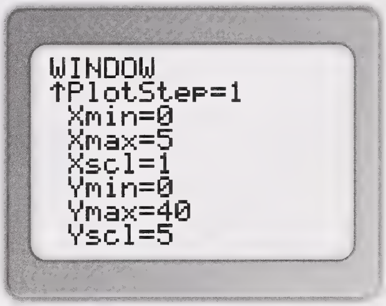


```
  n   | u(n) |
  1   | 1    |
  2   | 3    |
  3   | 7    |
  4   | 15   |
  5   | 31   |
  6   | 63   |
  7   | 127  |
n=1
```

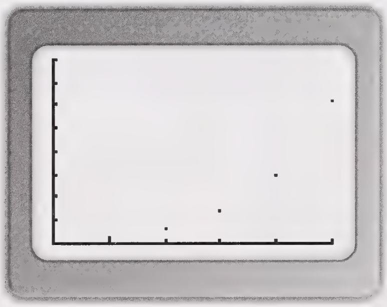
Press GRAPH to plot the graph.



```
WINDOW
 nMin=1
 nMax=10
 PlotStart=1
 PlotStep=1
 Xmin=0
 Xmax=5
↓Xscl=1
```



```
WINDOW
↑PlotStep=1
 Xmin=0
 Xmax=5
 Xscl=1
 Ymin=0
 Ymax=40
 Yscl=5
```



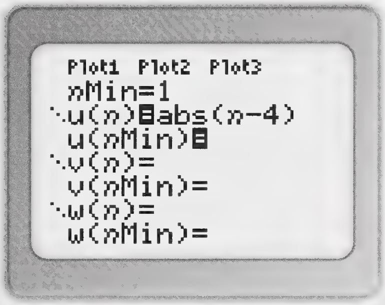
16. **a.**

$$\begin{aligned} t_n &= |n-4| \\ t_1 &= |1-4| \\ &= |-3| \\ &= 3 \end{aligned} \qquad \begin{aligned} t_n &= |n-4| \\ t_2 &= |2-4| \\ &= |-2| \\ &= 2 \end{aligned} \qquad \begin{aligned} t_n &= |n-4| \\ t_3 &= |3-4| \\ &= |-1| \\ &= 1 \end{aligned} \qquad \begin{aligned} t_n &= |n-4| \\ t_4 &= |4-4| \\ &= 0 \end{aligned} \qquad \begin{aligned} t_n &= |n-4| \\ t_5 &= |5-4| \\ &= 1 \end{aligned}$$

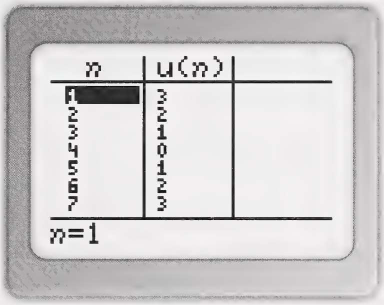
The first five terms are 3, 2, 1, 0, and 1.

b. To view the table of values, press the following:

Y= MATH [Select the NUM menu.] 1 X,T,Θ,*n* − 4) 2nd [TABLE]



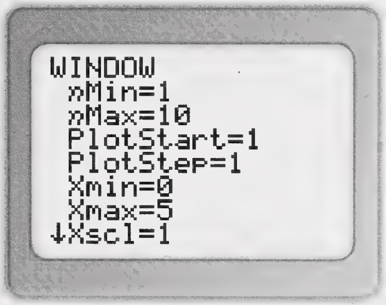
```
Plot1 Plot2 Plot3
 nMin=1
·.u(n)=abs(n-4)
 u(nMin)=
·.v(n)=
 v(nMin)=
·.w(n)=
 w(nMin)=
```



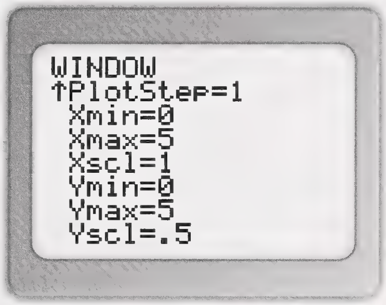
n	u(n)
1	3
2	2
3	1
4	0
5	1
6	2
7	3

n=1

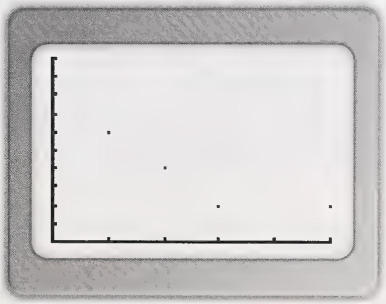
Press GRAPH to plot the graph.



```
WINDOW
 nMin=1
 nMax=10
 PlotStart=1
 PlotStep=1
 Xmin=0
 Xmax=5
↓Xscl=1
```



```
WINDOW
↑PlotStep=1
 Xmin=0
 Xmax=5
 Xscl=1
 Ymin=0
 Ymax=5
 Yscl=.5
```



Section 1: Activity 1 (continued)

19. The sequence is arithmetic, where the common difference, *d*, is 3 and the first term, *a*, is 1. Therefore, the next two terms are as follows:

$$1, 4, 7, 10, \mathbf{13}, \mathbf{16}, \ldots$$

$$\begin{aligned} \therefore\ t_n &= a+(n-1)d \\ &= 1+(n-1)3 \\ &= 1+3n-3 \\ &= 3n-2 \end{aligned}$$

20. $0.3,\ 0.09,\ 0.027,\ 0.0081, \ldots$

$0.3^1,\ 0.3^2,\ 0.3^3,\ 0.3^4, \ldots$

Each term is 0.3 times the preceding term. Therefore, the next two terms are as follows:

$$0.3^1,\ 0.3^2,\ 0.3^3,\ 0.3^4,\ \mathbf{0.3^5},\ \mathbf{0.3^6}, \ldots$$

$$0.3,\ 0.09,\ 0.027,\ 0.0081,\ \mathbf{0.002\,43},\ \mathbf{0.000\,729}, \ldots$$

Either answer here is correct.

$$\therefore\ t_n = 0.3^n$$

21. $2,\quad -4,\quad 8,\quad -16, \ldots$

$2,\ 2(-2)^1,\ 2(-2)^2,\ 2(-2)^3, \ldots$

Each term is -2 times the preceding term.

$$\therefore\ 2, -4, 8, -16, \mathbf{32}, \mathbf{-64}, \ldots$$

$$\begin{aligned} \therefore\ t_n &= 2(-2)^{n-1} \\ &= 2(-1)^{n-1}(2)^{n-1} \\ &= (-1)^{n-1}2^n \end{aligned}$$

b. Textbook question 26 of "Applications and Problem Solving," pp. 290 and 291

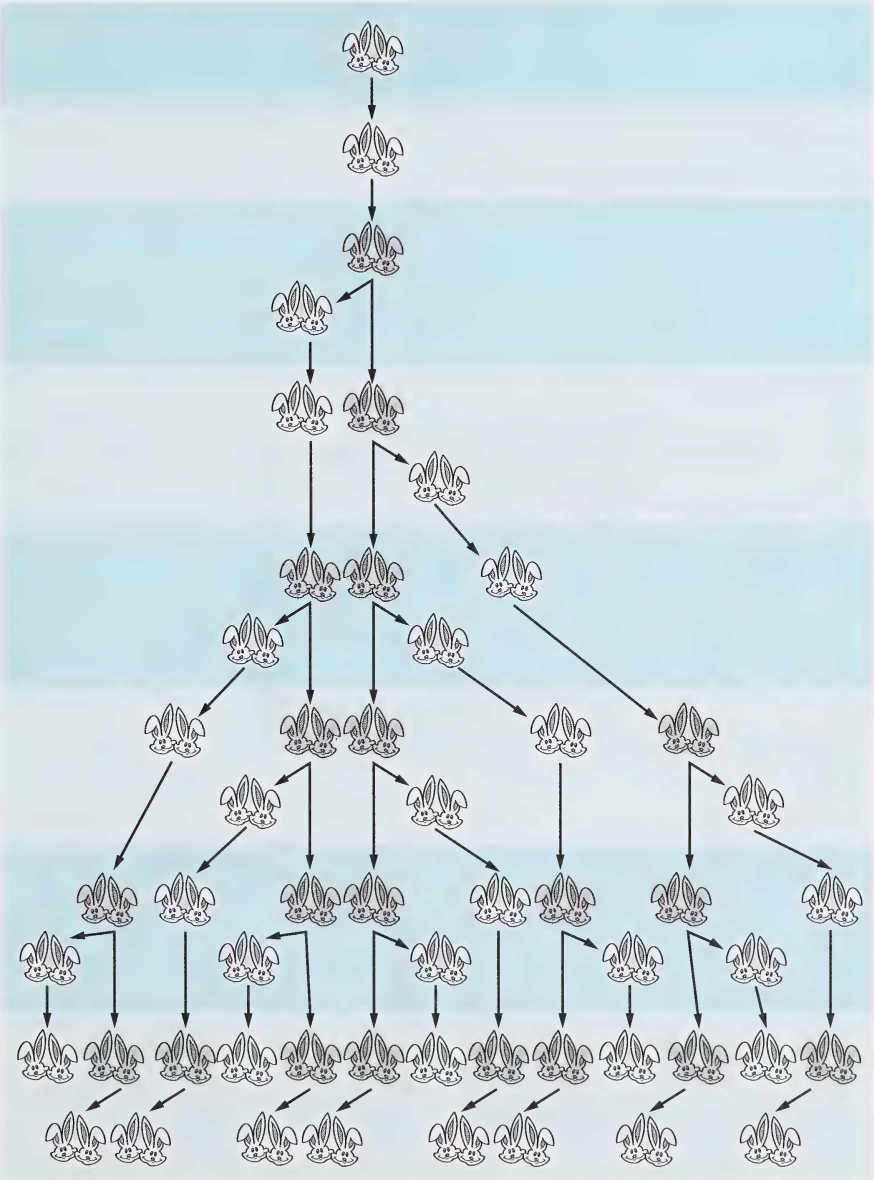
26. a. Each term is the sum of the two preceding terms.

$$\therefore\ 1, 1, 2, 3, 5, 8, 13, \mathbf{21}, \mathbf{34}, \mathbf{55}$$

$$\therefore\ t_n = t_{n-1} + t_{n-2}\text{, where } t_1 = 1 \text{ and } t_2 = 1$$

Note: The recursive formula is dependent on two terms; therefore, the first two terms must be stated.

b. Let represent a mature pair of rabbits, and let represent an immature pair of rabbits.



This sequence assumes that each mature pair of rabbits bear a male and female pair and that no rabbits die.

Section 1: Activity 1 (continued)

3. a. **Textbook question "Explore: Use a Pattern," p. 292**

Week 1	Week 2	Week 3	Week 4	Week 5	Week 6	Week 7
9	12	15	18	21	24	27

The sequence 9, 12, 15, 18, 21, 24, 27 represents how far Melanie runs each training day for the first seven weeks.

b. **Textbook questions 1 to 4 of "Inquire," p. 292**

1. Each term is determined by adding 3 to the preceding term.

$\therefore t_n = t_{n-1} + 3$, where $t_1 = 9$ and $2 \le n \le 7$, $n \in \mathrm{N}$

The first term must be part of the formula.

2. The sequence is arithmetic.

Using the pattern shown, the sequence $9,\ 9+3(1),\ 9+3(2),\ 9+3(3),\ 9+3(4),\ 9+3(5),$ $9+3(6)$ can be written as $t_n = 9+3(n-1)$.

3. **Method 1**

Continue writing the terms of the sequence until 42 is reached.

9, 12, 15, 18, 21, 24, 27, 30, 33, 36, 39, **42**

Melanie will run 42 km in week 12.

Method 2

Substitute $t_n = 42$ into the explicit formula, and solve for *n*.

$$\begin{aligned} t_n &= 9+3(n-1) \\ 42 &= 9+3(n-1) \\ 33 &= 3(n-1) \\ 11 &= n-1 \\ 12 &= n \\ n &= 12 \end{aligned}$$

Melanie will run 42 km in week 12.

4. The sum of the first 10 terms will give you the total distance Melanie ran if she ran once per week for the first 10 weeks.

   If you wanted to obtain the total distance Melanie ran during the first 10 weeks of her training, multiply the sum of the first 10 terms by 5.

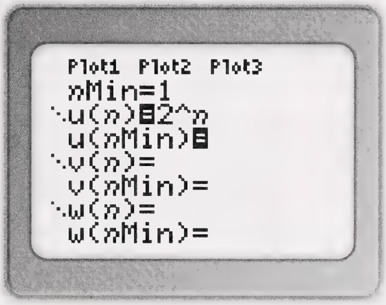
4. a. Textbook questions 3, 5, 11, 12, 18, and 20 of "Practice," p. 295

3. The sequence 2, 4, 8, 16, 32, … is **not** arithmetic.

   The sequence can be rewritten as 2^1, 2^2, 2^3, 2^4, 2^5, … . Therefore, the explicit formula is $t_n = 2^n$.

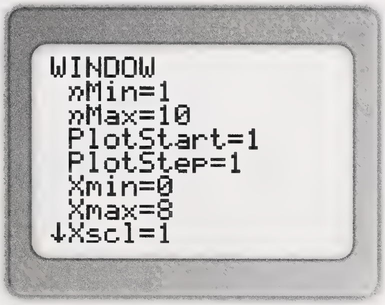
   Use the formula to plot the terms of the sequence.

   Y= 2 ^ X,T,Θ,*n*

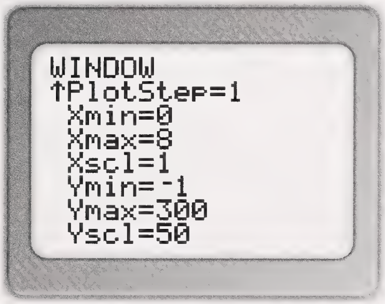


```
Plot1 Plot2 Plot3
 nMin=1
\u(n)=2^n
 u(nMin)=
\v(n)=
 v(nMin)=
\w(n)=
 w(nMin)=
```

Use the following window settings.



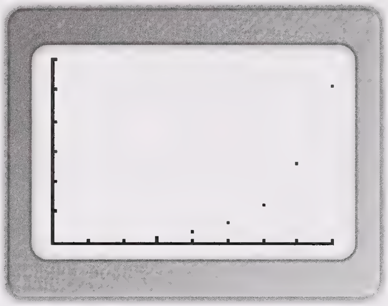
```
WINDOW
 nMin=1
 nMax=10
 PlotStart=1
 PlotStep=1
 Xmin=0
 Xmax=8
↓Xscl=1
```



```
WINDOW
↑PlotStep=1
 Xmin=0
 Xmax=8
 Xscl=1
 Ymin=-1
 Ymax=300
 Yscl=50
```

Section 1: Activity 1 (continued)

Press GRAPH to plot the sequence.



The terms do not lie on a straight line.

5. The sequence $-2, -5, -8, -11, \ldots$ is arithmetic. For this sequence, $a = -2$ and $d = -3$.

11. $t_2 = -12$ and $t_5 = 9$

$$\begin{aligned} \therefore\ t_n &= a + (n-1)d \\ t_2 &= a + (2-1)d \\ -12 &= a + d \quad ① \end{aligned}$$

$$\begin{aligned} \therefore\ t_n &= a + (n-1)d \\ t_5 &= a + (5-1)d \\ 9 &= a + 4d \quad ② \end{aligned}$$

Solve the system of equations.

$$\begin{aligned} -12 &= a + d & ① \\ 9 &= a + 4d & ② \\ \hline -21 &= -3d & ① - ② \\ 7 &= d & \end{aligned}$$

Substitute into $d = 7$ into ①.

$$\begin{aligned} -12 &= a + d \\ -12 &= a + 7 \\ a &= -19 \end{aligned}$$

$$\begin{aligned} \therefore t_n &= a + (n-1)d \\ &= -19 + (n-1)7 \\ &= -19 + 7n - 7 \\ &= 7n - 26 \end{aligned}$$

Check

If $n = 2$, then

$$\begin{aligned} t_n &= 7n - 26 \\ t_2 &= 7(2) - 26 \\ &= 14 - 26 \\ &= -12 \end{aligned}$$

If $n = 5$, then

$$\begin{aligned} t_n &= 7n - 26 \\ t_n &= 7n - 26 \\ t_5 &= 7(5) - 26 \\ &= 35 - 26 \\ &= 9 \end{aligned}$$

12. $t_7 = 37$ and $t_{10} = 22$

$$\begin{aligned} \therefore t_n &= a + (n-1)d \\ t_7 &= a + (7-1)d \\ 37 &= a + 6d \quad ① \end{aligned}$$

$$\begin{aligned} \therefore t_n &= a + (n-1)d \\ t_{10} &= a + (10-1)d \\ 22 &= a + 9d \quad ② \end{aligned}$$

Solve the system of equations.

$$\begin{array}{rll} 37 = a + 6d & & ① \\ 22 = a + 9d & & ② \\ \hline 15 = \quad -3d & & ① - ② \\ -5 = d & & \end{array}$$

Section 1: Activity 1 (continued)

Substitute $d=-5$ into ①.

$$\begin{aligned}37&=a+6d\\37&=a+6(-5)\\37&=a-30\\a&=67\end{aligned}$$

$$\begin{aligned}\therefore\ t_n&=a+(n-1)d\\&=67+(n-1)(-5)\\&=67-5n+5\\&=72-5n\end{aligned}$$

Check

If $n=7$, then

$$\begin{aligned}t_n&=72-5n\\t_7&=72-5(7)\\&=72-35\\&=37\end{aligned}$$

If $n=10$, then

$$\begin{aligned}t_n&=72-5n\\t_{10}&=72-5(10)\\&=72-50\\&=22\end{aligned}$$

18. $61,\ 55,\ 49,\ \ldots,\ -119$

$a=61,\ d=-6,\ t_n=-119$

$$\begin{aligned}\therefore\ t_n&=a+(n-1)d\\-119&=61+(n-1)(-6)\\-119&=61-6n+6\\-119&=67-6n\\-186&=-6n\\6n&=186\\n&=31\end{aligned}$$

There are 31 terms in the sequence.

To calculate the sum of the terms of the sequence, use

$$\begin{aligned} S_n &= \frac{n}{2}(a+t_n) \\ S_{31} &= \frac{31}{2}\left[61+(-119)\right] \\ &= \frac{31}{\underset{1}{\cancel{2}}}\left(\overset{-29}{\cancel{-58}}\right) \\ &= -899 \end{aligned}$$

20. $1+2k,\ 2+4k,\ 3+6k,\ \ldots,\ 19+38k$

$a = 1+2k$, $d = (2+4k)-(1+2k)$, and $t_n = 19+38k$

$$\begin{aligned} &= 2+4k-1-2k \\ &= 1+2k \end{aligned}$$

$$\begin{aligned} \therefore\ t_n &= a+(n-1)d \\ 19+38k &= (1+2k)+(n-1)(1+2k) \\ 19+38k &= \cancel{(1+2k)}+(1+2k)n-\cancel{(1+2k)} \\ 19+38k &= (1+2k)n \\ n &= \frac{19+38k}{(1+2k)} \\ &= \frac{19\cancel{(1+2k)}}{\cancel{(1+2k)}} \\ &= 19 \end{aligned}$$

To calculate the sum of the terms of the sequence, use

$$\begin{aligned} S_n &= \frac{n}{2}(a+t_n) \\ S_{19} &= \frac{19}{2}(1+2k+19+38k) \\ &= \frac{19}{2}(20+40k) \\ &= \frac{19}{\cancel{2}}(\cancel{2})(10+20k) \\ &= 190+380k \end{aligned}$$

Section 1: Activity 1 (continued)

b. Textbook questions 33, 35.a., 36.a., 40, and 41 of "Applications and Problem Solving," p. 296

33.
$$\begin{aligned} a &= t_1 \\ &= 5(1)+4 \\ &= 9 \end{aligned}$$

$$\begin{aligned} d &= t_n - t_{n-1} \\ &= (5n+4)-[5(n-1)+4] \\ &= 5n+4-(5n-5+4) \\ &= 5n+4-(5n-1) \\ &= 5n+4-5n+1 \\ &= 5 \end{aligned}$$

35. a. $a = 1000$, $d = 150$, and $n = 12$

$$\begin{aligned} \therefore\ t_n &= a+(n-1)d \\ t_{12} &= 1000+(12-1)(150) \\ &= 1000+11(150) \\ &= 1000+1650 \\ &= 2650 \end{aligned}$$

The apprentice's salary for the last month of the year was $2650.

36. a. Multiples of 6 must be between 65 and 391.

$\therefore\ a = 66$, $d = 6$, and $t_n = 390$

$$\begin{aligned} \therefore\ t_n &= a+(n-1)d \\ 390 &= 66+(n-1)6 \\ 390 &= 66+6n-6 \\ 390 &= 6n+60 \\ 330 &= 6n \\ 6n &= 330 \\ n &= 55 \end{aligned}$$

There are 55 multiples of 6 between 65 and 391.

40. An arithmetic sequence is as follows:

$$a,\ a+d,\ a+2d,\ \dots,\ a+(n-1)d$$

If the new sequence is multiplied by a constant, *k*, then the sequence becomes

$$ka,\ k(a+d),\ k(a+2d),\ \dots,\ k[a+(n-1)d]$$

To show it is arithmetic you need to calculate $t_n - t_{n-1}$. If it is a constant, the sequence is arithmetic.

$$t_n = k[a+(n-1)d] \text{ and } t_{n-1} = k[a+(n-1-1)d]$$

$$\begin{aligned} \therefore\ t_n - t_{n-1} &= k[a+(n-1)d]-k[a+(n-2)d] \\ &= k(a+dn-d)-k(a+dn-2d) \\ &= ka+kdn-kd-ka-kdn+2kd \\ &= -kd+2kd \\ &= kd \end{aligned}$$

The new sequence is arithmetic, and has a common difference of *kd*.

41. **a.** Sequence 1

$$t_3 = 16 \text{ and } t_{10} = 100$$

$$\begin{aligned} \therefore\ t_n &= a+(n-1)d \\ t_3 &= a+(3-1)d \\ 16 &= a+2d \quad ① \end{aligned}$$

$$\begin{aligned} \therefore\ t_n &= a+(n-1)d \\ t_{10} &= a+(10-1)d \\ 100 &= a+9d \quad ② \end{aligned}$$

Solve the system of equations.

$$\begin{array}{rl} 16 = a+2d & ① \\ \underline{100 = a+9d} & ② \\ -84 = \quad -7d & ① - ② \\ 7d = 84 & \\ d = 12 & \end{array}$$

Section 1: Activity 1 (continued)

Substitute $d=12$ into ①.

$$\begin{aligned} 16 &= a+2d \\ 16 &= a+2(12) \\ 16 &= a+24 \\ a &= -8 \end{aligned}$$

$$\begin{aligned} \therefore\ t_n &= a+(n-1)d \\ &= -8+(n-1)(12) \\ &= -8+12n-12 \\ &= 12n-20 \end{aligned}$$

Sequence 2

$t_3=44$ and $t_{10}=100$

$$\begin{aligned} \therefore\ t_n &= a+(n-1)d \\ t_3 &= a+(3-1)d \\ 44 &= a+2d \quad ① \end{aligned}$$

$$\begin{aligned} \therefore\ t_n &= a+(n-1)d \\ t_{10} &= a+(10-1)d \\ 100 &= a+9d \quad ② \end{aligned}$$

Solve the system of equations.

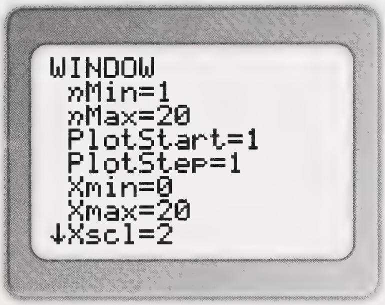
$$\begin{aligned} 44 &= a+2d \quad ① \\ 100 &= a+9d \quad ② \\ \hline -56 &= \quad -7d \quad ① - ② \\ 7d &= 56 \\ d &= 8 \end{aligned}$$

Substitute $d = 8$ into ①.

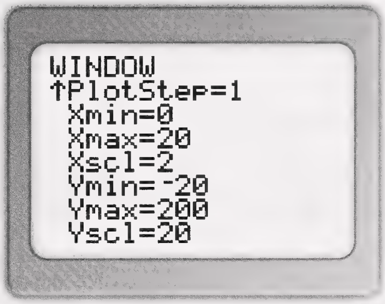
$$44 = a + 2d$$
$$44 = a + 2(8)$$
$$44 = a + 16$$
$$a = 28$$

$$\therefore t_n = a + (n-1)d$$
$$= 28 + (n-1)8$$
$$= 28 + 8n - 8$$
$$= 8n + 20$$

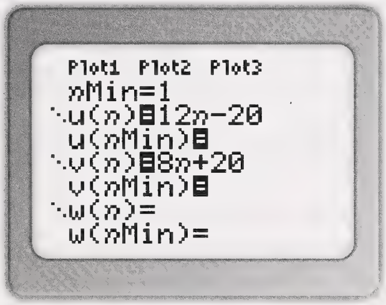
b.



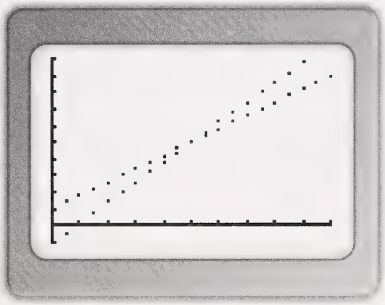
```
WINDOW
 nMin=1
 nMax=20
 PlotStart=1
 PlotStep=1
 Xmin=0
 Xmax=20
↓Xscl=2
```



```
WINDOW
↑PlotStep=1
 Xmin=0
 Xmax=20
 Xscl=2
 Ymin=-20
 Ymax=200
 Yscl=20
```



```
 Plot1 Plot2 Plot3
 nMin=1
·.u(n)=12n-20
 u(nMin)=
·.v(n)=8n+20
 v(nMin)=
·.w(n)=
 w(nMin)=
```



Both graphs consist of points that lie on a straight line. The difference between the two graphs is the slope. The slope of Sequence 1 is steeper than the slope of Sequence 2.

Section 1: Activity 1 (continued)

5. Textbook question 29 of "Applications and Problem Solving," p. 291

29. a.

Polygon	Number of Sides	Number of Diagonals
Triangle	3	0
Quadrilateral	4	2
Pentagon	5	5
Hexagon	6	9
Heptagon	7	14

Number of diagonals $= \frac{n(n-1)}{2} - n$, where n is the number of sides in the polygon

b. A one-dollar coin has 11 sides.

$$
\begin{aligned}
\therefore \text{ Number of diagonals} &= \frac{n(n-1)}{2} - n \\
&= \frac{11(11-1)}{2} - 11 \\
&= \frac{11\left(\overset{5}{10}\right)}{\underset{1}{2}} - 11 \\
&= 55 - 11 \\
&= 44
\end{aligned}
$$

Section 1: Activity 2

1. a. **Textbook question "Explore: Use Modelling," p. 297**

Number of Times Folded	Number of Layers of Paper
0	1
1	2
2	4
3	8
4	16

b. **Textbook questions 1 to 4 of "Inquire," p. 297**

1. The original size of the paper and the thickness of the paper affects the number of times you can fold it. There is a limit to the number of times you can fold the paper because the folded paper becomes too thick.

2. The expression for the number of layers after the paper is folded n times is $t_n = 1(2)^n$, where $n \in N$.

   The expression for the $(n+1)$th term is $t_{n+1} = 2^{n+1}$.

3. If you start with 3 sheets of paper, the number of layers after each of the first four foldings is as follows:

   6, 12, 24, 48

   The nth term is $t_n = 3(2)^n$, where $n \in N$.

   Each successive term is found by doubling the previous term.

Section 1: Activity 2 (continued)

4. The area of the original sheet is 1 square unit. After one fold, the area is $\frac{1}{2}$ square units. After 2 folds, the area is $\frac{1}{4}$ square units. The sequence representing the area is $1, \frac{1}{2}, \frac{1}{4}, \ldots$.

The multiplying factor is $\frac{1}{2}$.

Therefore, the expression for t_n is $t_n = \left(\frac{1}{2}\right)^n$, where n is the number of folds.

2. a. Textbook questions 1, 4, 10, 15, 17, and 22 of "Practice," p. 300

1. $a = 15$ and $r = \frac{5}{15} = \frac{1}{3}$

$15, 5, \frac{5}{3}, \mathbf{\frac{5}{9}}, \mathbf{\frac{5}{27}}, \ldots$

$\therefore\ t_n = 15\left(\frac{1}{3}\right)^{n-1}$

4. $a = 0.11$ and $r = \frac{0.33}{0.11} = 3$

0.11, 0.33, 0.99, **2.97**, **8.91**, …

$\therefore\ t_n = 0.11(3)^{n-1}$

10. $a = -15$ and $r = \frac{225}{-15} = -15$

-15, 225, -3375, **50 625**, **−759 375**, …

$\therefore\ t_n = -15(-15)^{n-1}$ or $t_n = (-15)^n$

15. $a = \dfrac{1}{5}$ and $r = \dfrac{\frac{3}{5}}{\frac{1}{5}}$

$$= \frac{3}{\cancel{5}} \times \frac{\cancel{5}}{1}$$

$$= 3$$

$$\therefore\ t_n = ar^{n-1}$$

$$= \frac{1}{5}(3)^{n-1}$$

$$t_9 = \frac{1}{5}(3)^{9-1}$$

$$= \frac{1}{5}(3)^{8}$$

$$= \frac{6561}{5} \text{ or } 1312.2$$

17. $t_3 = 10$ and $t_6 = 80$

$$\therefore\ t_n = ar^{n-1}$$

$$t_3 = ar^{3-1}$$

$$10 = ar^{2} \quad ①$$

$$\therefore\ t_n = ar^{n-1}$$

$$t_6 = ar^{6-1}$$

$$80 = ar^{5} \quad ②$$

Divide ② by ①.

$$\frac{80}{10} = \frac{ar^5}{ar^2}$$

$$8 = r^3$$

$$r^3 = 8$$

$$r = 2$$

Substitute $r = 2$ into ①.

$$10 = ar^2$$

$$10 = a(2)^2$$

$$10 = 4a$$

$$4a = 10$$

$$a = \frac{10}{4}$$

$$= \frac{5}{2}$$

$$\therefore\ t_n = ar^{n-1}$$

$$= \frac{5}{2}(2)^{n-1}$$

$$t_8 = \frac{5}{2}(2)^{8-1}$$

$$= \frac{5}{2}(2)^{7}$$

$$= 320$$

Section 1: Activity 2 (continued)

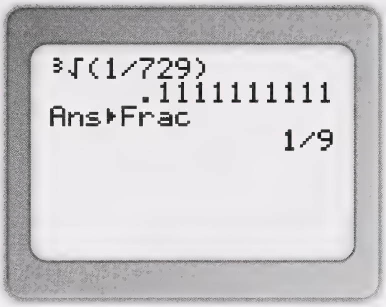
22. $t_3 = 59\,049$ and $t_6 = 81$

$$\therefore\ t_n = ar^{n-1}$$
$$t_3 = ar^{3-1}$$
$$59\,049 = ar^2 \quad ①$$

$$\therefore\ t_n = ar^{n-1}$$
$$t_6 = ar^{6-1}$$
$$81 = ar^5 \quad ②$$

Divide ② by ①.

$$\frac{81}{59\,049} = \frac{ar^5}{ar^2}$$
$$\frac{1}{729} = r^3$$
$$r^3 = \sqrt[3]{\frac{1}{729}}$$
$$= \frac{1}{9}$$



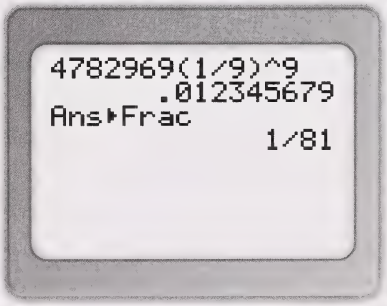
```
³√(1/729)
        .1111111111
Ans▸Frac
                1/9
```

Note: *To change a decimal value to a fraction, press* MATH 1 ENTER.

Substitute $r = \frac{1}{9}$ into ①.

$$59\,049 = ar^2$$
$$59\,049 = a\left(\frac{1}{9}\right)^2$$
$$59\,049 = a\left(\frac{1}{81}\right)$$
$$a = 4\,782\,969$$

$$\therefore\ t_n = ar^{n-1}$$
$$= 4\,782\,969\left(\frac{1}{9}\right)^{n-1}$$
$$t_{10} = 4\,782\,969\left(\frac{1}{9}\right)^{10-1}$$
$$= 4\,782\,969\left(\frac{1}{9}\right)^{9}$$
$$= \frac{1}{81}$$



```
4782969(1/9)^9
         .012345679
Ans▸Frac
               1/81
```

b. **Textbook questions 23, 24, 27, 29, 31, 33, and 35 of "Applications and Problem Solving," pp. 300 and 301**

23. **a.** $x, 6, 18, y$

$$\begin{aligned} r &= \frac{18}{6} \\ &= 3 \end{aligned}$$

$$\therefore\ x = \frac{6}{3} = 2 \quad \text{and} \quad y = 18 \times 3 = 54$$

b. $6, 18, x, y$

$$\begin{aligned} r &= \frac{18}{6} \\ &= 3 \end{aligned}$$

$$\therefore\ x = 18 \times 3 = 54 \quad \text{and} \quad y = 54 \times 3 = 162$$

c. $6, x, 18, y$

$$\begin{aligned} 6 \times r \times r &= 18 \\ 6r^2 &= 18 \\ r^2 &= 3 \\ r &= \pm\sqrt{3} \end{aligned}$$

For $r = \sqrt{3}$, $x = 6\sqrt{3}$ and $y = 18\sqrt{3}$. For $r = -\sqrt{3}$, $x = -6\sqrt{3}$ and $y = -18\sqrt{3}$.

d. $6, x, y, 18$

$$\begin{aligned} 6 \times r \times r \times r &= 18 \\ 6r^3 &= 18 \\ r^3 &= 3 \\ r &= \sqrt[3]{3} \end{aligned}$$

$$\therefore\ x = 6\sqrt[3]{3} \quad \text{and} \quad \begin{aligned} y &= 6\sqrt[3]{3} \times \sqrt[3]{3} \\ &= 6\sqrt[3]{9} \end{aligned}$$

Section 1: Activity 2 (continued)

24. $t_3 = 2$ and $t_5 = 4$

$$\begin{aligned} \therefore\ t_n &= ar^{n-1} \\ t_3 &= ar^{3-1} \\ 2 &= ar^2 \quad ① \end{aligned}$$

$$\begin{aligned} t_n &= ar^{n-1} \\ t_5 &= ar^{5-1} \\ 4 &= ar^4 \quad ② \end{aligned}$$

Divide ② by ①.

$$\begin{aligned} \frac{4}{2} &= \frac{ar^4}{ar^2} \\ 2 &= r^2 \\ r &= \sqrt{2} \end{aligned}$$

The value of r cannot be negative.

Substitute $r = \sqrt{2}$ into ①.

$$\begin{aligned} 2 &= ar^2 \\ 2 &= a\left(\sqrt{2}\right)^2 \\ 2 &= 2a \\ a &= 1 \end{aligned}$$

$$\begin{aligned} \therefore\ t_n &= ar^{n-1} \\ &= 1\left(\sqrt{2}\right)^{n-1} \\ t_{10} &= \left(\sqrt{2}\right)^{10-1} \\ &= \left(\sqrt{2}\right)^9 \\ &= 16\sqrt{2} \\ &\doteq 22.6 \end{aligned}$$

$$\begin{aligned} \left(\sqrt{2}\right)^9 &= \left(2^{\frac{1}{2}}\right)^9 \\ &= 2^{4\frac{1}{2}} \\ &= 2^4 \times 2^{\frac{1}{2}} \\ &= 16\sqrt{2} \end{aligned}$$

The number associated with the tenth *f*-stop is approximately 22.6.

27. Because the annual depreciation is 30%, the value of the computer is 70% of the previous year's value. Therefore, the value of the computer each year is as follows:

$$\underset{\substack{\uparrow\\\text{now}}}{3000},\ \overset{\substack{\text{after 1 year}\\\downarrow}}{3000(0.7)},\ \underset{\substack{\uparrow\\\text{after 2 years}}}{3000(0.7)^2},\ \overset{\substack{\text{after 3 years}\\\downarrow}}{3000(0.7)^3},\ \underset{\substack{\uparrow\\\text{after 4 years}}}{3000(0.7)^4},\ \ldots$$

$$\begin{aligned}\therefore\ \text{Value after 4 years} &= 3000(0.7)^4\\ &= 720.30\end{aligned}$$

After 4 years, Revenue Canada would assign a value of $720.30 to the computer.

29. Let P be the original size of the image.

After 1 reduction, the size of the image is $P(0.64)$. After 2 reductions, the size of the image is $P(0.64)^2$. After n reductions, where $n \in W$, the size of the image is $P(0.64)^n$.

$$\begin{aligned}\therefore\ P(0.64)^n &\le 0.10P\\ 0.64^n &\le 0.10\end{aligned}$$

Solve for n.

Method 1: Using Guess and Check

Try $n = 5$.

LS	RS
0.64^n	0.10
$= 0.64^5$	
$\doteq 0.107$	

LS $\not\le$ RS

Try $n = 6$.

LS	RS
0.64^n	0.10
$= 0.64^6$	
$\doteq 0.069$	

LS $\le$ RS

Method 2: Using Logarithms

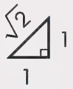
$$\begin{aligned}0.64^n &= 0.10\\ \log 0.64^n &= \log 0.10\\ n \log 0.64 &= \log 0.10\\ n &= \frac{\log 0.10}{\log 0.64}\\ &\doteq 5.16\end{aligned}$$

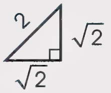
Since $n \in W$, $n = 6$.

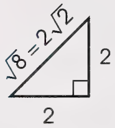
It will take 6 reductions to reduce the image to less than 10% of its original size.

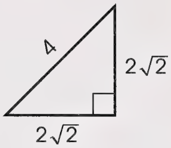
Section 1: Activity 2 (continued)

31. a.









To show that the lengths of the hypotenuses of right triangles constructed by continuing this pattern form a geometric sequence, you need to show $\frac{t_{n+1}}{t_n}=r$, where r is a constant.

If the equal sides of the right triangle have length x, then the hypotenuse is $\sqrt{x^2+x^2}=\sqrt{2}x$.

The length of the next hypotenuse is $\sqrt{(\sqrt{2}x)^2+(\sqrt{2}x)^2}=\sqrt{4x^2}=2x$. The ratio is $\frac{2x}{\sqrt{2}x}=\sqrt{2}$, which is a constant.

b.

$$\begin{aligned} t_n &= ar^{n-1} \\ &= \sqrt{2}(\sqrt{2})^{n-1} \\ &= (\sqrt{2})^n \\ t_7 &= (\sqrt{2})^7 \\ &= 8\sqrt{2} \end{aligned}$$

The length of the hypotenuse of the seventh triangle is $8\sqrt{2}$ units.

33. a. The terms 5, 9, 13, 17 form an arithmetic sequence where the common difference, d, is 4.

The terms 2^5, 2^9, 2^{13}, 2^{17} form a geometric sequence where the common ratio, r, is 2^4 or 16.

b. The terms n, p, q, r form an arithmetic sequence. If k is the common difference, the sequence can be written as n, $n+k$, $n+2k$, $n+3k$.

Now, the sequence x^n, x^p, x^q, x^r can be written as x^n, x^{n+k}, x^{n+2k}, x^{n+3k}, which is equal to x^n, $x^n x^k$, $x^n x^{2k}$, $x^n x^{3k}$.

Notice that each term in the sequence is x^k times the previous term. Therefore, the sequence is geometric.

35.

$$\begin{aligned} t_1 + t_2 + t_3 &= 21 \\ ar^{1-1} + ar^{2-1} + ar^{3-1} &= 21 \\ a + ar + ar^2 &= 21 \\ a(1+r+r^2) &= 21 \quad ① \end{aligned}$$

$$\begin{aligned} t_4 + t_5 + t_6 &= 168 \\ ar^{4-1} + ar^{5-1} + ar^{6-1} &= 168 \\ ar^3 + ar^4 + ar^5 &= 168 \\ ar^3(1+r+r^2) &= 168 \quad ② \end{aligned}$$

Divide ② by ①.

$$\begin{aligned} \frac{ar^3(1+r+r^2)}{a(1+r+r^2)} &= \frac{168}{21} \\ r^3 &= 8 \\ r &= 2 \end{aligned}$$

Substitute $r = 2$ into ①.

$$\begin{aligned} a(1+r+r^2) &= 21 \\ a(1+2+2^2) &= 21 \\ a(1+2+4) &= 21 \\ 7a &= 21 \\ a &= 3 \end{aligned}$$

The first three terms of the geometric sequence are 3, 6, and 12.

3. Textbook question “LOGIC POWER,” p. 305

To make it easier to describe your reasoning, label the columns, rows, and vacant cells like the following.

	[illegible]	[illegible]	[illegible]	[illegible]
Row 1	1	2	3	4
Row 2	5	H	7	8
Row 3	9	10	O	12
Row 4	13	14	15	16

Section 1: Activity 2 (continued)

The letter C occupies a corner square and cannot appear in row 1 because of the letter B and cannot appear in column 4 because of the letter D. Therefore, the letter C must be in cell 13.

1	2	3	4
5	H	7	8
9	10	O	12
C	14	15	16

Note: The remaining letters are A, B, D, E, F, G, I, J, K, L, M, N, and P.

The letter I cannot appear in column 2 or row 2 because of the letter H, cannot appear in column 3 or row 3 because of the letter O, and cannot appear in column 4 because of the letter J. Therefore, the letter I must be in cell 1.

I	2	3	4
5	H	7	8
9	10	O	12
C	14	15	16

Note: The remaining letters are A, B, D, E, F, G, J, K, L, M, N, and P.

The letter E cannot appear in column 1 or row 1 because of the letter I, cannot appear in column 3 or row 3 because of the letter O, and cannot appear in column 4 because of the letter D. Therefore, the letter E must be in cell 14.

I	2	3	4
5	H	7	8
9	10	O	12
C	E	15	16

Note: The remaining letters are A, B, D, F, G, J, K, L, M, N, and P.

The letter A cannot appear in column 1 or row 1 because of the letter I, cannot appear in column 2 or row 4 because of the letter E, and cannot appear in column 3 or row 3 because of the letter O. Therefore, the letter A must be in cell 8.

The letter K appears in the same row as the letter E, but cannot appear in column 4 because of the letter J. Therefore, K must be in cell 15.

I	2	3	4
5	H	7	A
9	10	O	12
C	E	K	16

Note: The remaining letters are B, D, F, G, J, L, M, N, and P.

The letter J is in column 4, but cannot appear in row 1 because of the letter I nor in row 4 because of the letter K. Therefore, the letter J must be in cell 12.

The letter D is in column 4, but cannot appear in row 4 because of letters C and E. Therefore, the letter D must be in cell 4.

The letter M occupies a corner square. Therefore, the letter M must be in cell 16.

The letter P does not appear in row 1 and cannot appear in column 3 or row 3 because of the letter O. Therefore, the letter P must be in cell 5.

I	2	3	D
P	H	7	A
9	10	O	J
C	E	K	M

Note: The remaining letters are B, F, G, L and N.

Section 1: Activity 2 (continued)

The letter N cannot appear in column 3 or row 3 because of the letter O. Therefore, the letter N must be in cell 2.

The letter B is in row 1. Therefore, it must be in cell 3.

The letter F cannot appear in cell 10 because of the letter E; and the letter G cannot appear in cell 10 because of the letter H. Therefore, the letter L must be in cell 10.

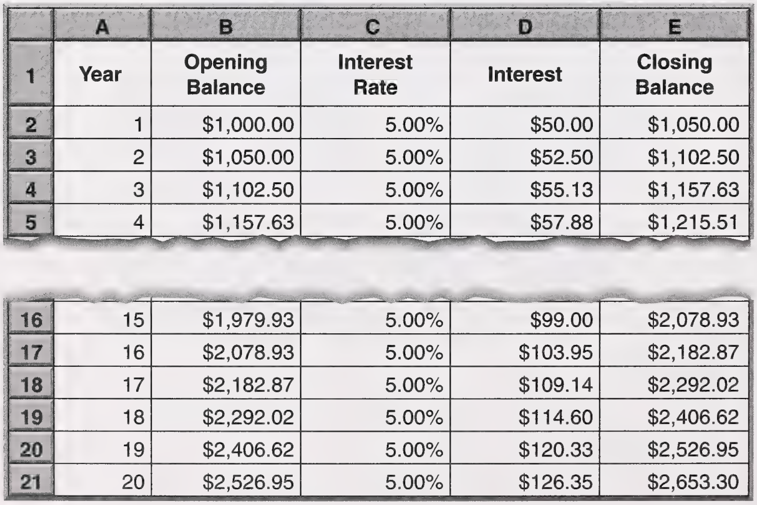
The letter G cannot appear in cell 7 because of the letter H. Therefore, the letter F must be in cell 7 and the letter G must be in cell 9.

I	N	B	D
P	H	F	A
G	L	O	J
C	E	K	M

Section 1: Activity 3

1. a. **Textbook question "Explore: Use a Spreadsheet," p. 302**

Setups may vary. **Note:** To save space, the following spreadsheet does not show rows 6 to 15.



	A	B	C	D	E
1	Year	Opening Balance	Interest Rate	Interest	Closing Balance
2	1	$1,000.00	5.00%	$50.00	$1,050.00
3	2	$1,050.00	5.00%	$52.50	$1,102.50
4	3	$1,102.50	5.00%	$55.13	$1,157.63
5	4	$1,157.63	5.00%	$57.88	$1,215.51
16	15	$1,979.93	5.00%	$99.00	$2,078.93
17	16	$2,078.93	5.00%	$103.95	$2,182.87
18	17	$2,182.87	5.00%	$109.14	$2,292.02
19	18	$2,292.02	5.00%	$114.60	$2,406.62
20	19	$2,406.62	5.00%	$120.33	$2,526.95
21	20	$2,526.95	5.00%	$126.35	$2,653.30

b. Textbook questions 1 to 3 of "Inquire," p. 302

1. At the end of 20 years, the investment has grown from $1000 to $2653.30, a total of $1653.30.

   The original amount doubled after 15 years.

2. The amounts at the end of each year form a geometric sequence because the interest is also invested each year (compound interest).

   The common ratio is 1.05. If the interest rate was 7%, the common ratio would be 1.07.

3. $A = P(1+i)^n$

2. a. Textbook questions 1, 3, 5, 7, 9, 11, 15, 17, and 19 of "Practice," p. 304

1.
$$\begin{aligned} i &= \frac{10\%}{2} \\ &= 5\% \end{aligned}$$

3.
$$\begin{aligned} i &= \frac{10\%}{4} \\ &= 2.5\% \end{aligned}$$

5.
$$\begin{aligned} i &= \frac{12\%}{12} \\ &= 1\% \end{aligned}$$

7.
$$\begin{aligned} i &= \frac{3\frac{3}{4}\%}{2} \\ &= \frac{\frac{15}{4}\%}{2} \\ &= \frac{15}{8}\% \\ &= 1\frac{7}{8}\% \end{aligned}$$

or

$$\begin{aligned} i &= \frac{3\frac{3}{4}\%}{2} \\ &= \frac{3.75\%}{2} \\ &= 1.875\% \\ &= 1\frac{7}{8}\% \end{aligned}$$

9.
$$\begin{aligned} i &= \frac{5.5\%}{2} \\ &= 2.75\% \end{aligned}$$

11.
$$\begin{aligned} n &= 2 \times 5 \\ &= 10 \end{aligned}$$

Section 1: Activity 3 (continued)

15. $n = 12 \times 6$

$= 72$

17. $P = 5000$

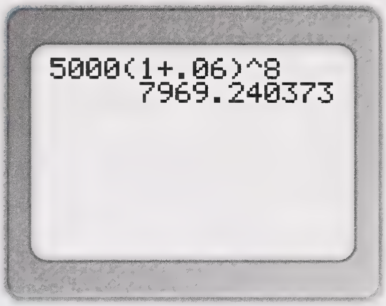
$i = 6\% = 0.06$

$n = 8$

$\therefore\ A = P(1+i)^n$

$= 5000(1+0.06)^8$

$= \$7969.24$



```
5000(1+.06)^8
       7969.240373
```

19. $P = 12\,000$

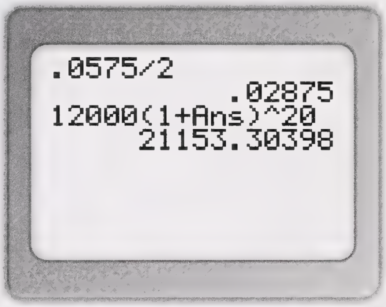
$i = \dfrac{5.75\%}{2} = 0.028\,75$

$n = 2 \times 10 = 20$

$\therefore\ A = P(1+i)^n$

$= 12\,000(1+0.028\,75)^{20}$

$= \$21\,153.30$



```
.0575/2
            .02875
12000(1+Ans)^20
       21153.30398
```

b. **Textbook questions 27 to 31 and 33 of “Applications and Problem Solving,” pp. 304 and 305**

27. **Option 1: Trust Company**

$P = 100\,000$

$i = \dfrac{6\%}{2} = 0.03$

$n = 2 \times 5 = 10$

$\therefore\ A = P(1+i)^n$

$= 100\,000(1+0.03)^{10}$

$= \$134\,391.64$

Option 2: Investment Broker

First 3 years,

$P = 100\,000$

$i = 7\% = 0.07$

$n = 3$

$\therefore\ A = P(1+i)^n$

$= 100\,000(1+0.07)^3$

$= \$122\,504.30$

Next 2 years,

$$\begin{aligned} P &= 122\ 504.30 \\ i &= 5\% = 0.05 \\ n &= 2 \end{aligned}$$

$$\begin{aligned} \therefore\ A &= P(1+i)^n \\ &= 122\ 504.30(1+0.05)^2 \\ &= \$135\ 060.99 \end{aligned}$$

The investment broker provides the greater return.

28. **First Investment**

$$\begin{aligned} P &= 1000 \\ i &= \frac{8\%}{2} = 0.04 \\ n &= 2 \times 2 = 4 \end{aligned}$$

$$\begin{aligned} \therefore\ A &= P(1+i)^n \\ &= 1000(1+0.04)^4 \\ &= \$1169.86 \end{aligned}$$

Second Investment

$$\begin{aligned} P &= 1000 \\ i &= \frac{9\%}{4} = 0.0225 \\ n &= 4 \end{aligned}$$

$$\begin{aligned} \therefore\ A &= P(1+i)^n \\ &= 1000(1+0.0225)^4 \\ &= \$1093.08 \end{aligned}$$

$$\begin{aligned} \therefore\ \text{Total} &= 1169.86 + 1093.08 \\ &= 2262.94 \end{aligned}$$

The investments will provide \$2262.94 for their holiday.

29. Determine the value of the investment today.

$$\begin{aligned} P &= 10\ 000 \\ i &= \frac{6\%}{2} = 0.03 \\ n &= 2 \times 5 = 10 \end{aligned}$$

$$\begin{aligned} \therefore\ A &= P(1+i)^n \\ &= 10\ 000(1+0.03)^{10} \\ &= 13\ 439.16 \end{aligned}$$

Determine the amount 5 years from now.

$$\begin{aligned} P &= 13\ 439.16 \\ i &= 7\% = 0.07 \\ n &= 5 \end{aligned}$$

$$\begin{aligned} \therefore\ A &= P(1+i)^n \\ &= 13\ 439.16(1+0.07)^5 \\ &= 18\ 849.12 \end{aligned}$$

In 5 years, the investment will be worth \$18 849.12.

Section 1: Activity 3 (continued)

30. **a.** $P = 500$, $i = \frac{12\%}{12} = 0.01$, and $n = 15$

$$\begin{aligned} \therefore\ A &= P(1+i)^n \\ &= 500(1+0.01)^{15} \\ &= 580.48 \end{aligned}$$

Farhana's bond is worth \$580.48 in 15 months.

b. $A = 650$, $P = 500$, and $i = \frac{12\%}{12} = 0.01$

$$\begin{aligned} \therefore\ A &= P(1+i)^n \\ 650 &= 500(1+0.01)^n \\ 1.3 &= 1.01^n \\ \log 1.3 &= \log 1.01^n \\ \log 1.3 &= n \log 1.01 \\ n &= \frac{\log 1.3}{\log 1.01} \\ &\doteq 26.4 \end{aligned}$$

Farhana's investment will be worth at least \$650 in 27 months.

31. **a.** $P = 3000$

$$\begin{aligned} i &= \frac{9\%}{2} = 0.045 \\ n &= 2 \times 3 \\ &= 6 \end{aligned}$$

$$\begin{aligned} \therefore\ A &= P(1+i)^n \\ &= 3000(1+0.045)^6 \\ &= 3906.78 \end{aligned}$$

The interest charged on the loan is \$906.78.

b. $P = 3000$

$i = 8\% = 0.08$

$n = 3$

$$\begin{aligned}\therefore\ A &= P(1+i)^n \\ &= 3000(1+0.08)^3 \\ &= 3779.14\end{aligned}$$

The RRSPs will be worth \$3779.14.

c. $$\begin{aligned}\text{Loss} &= 3906.78 - 3779.14 \\ &= 127.64\end{aligned}$$

Marla will take a loss of \$127.64.

d. This might be to Marla's advantage because she will have to pay less in taxes. She hopes that the amount she pays in taxes will be reduced by more than \$127.64.

33. $P = 10\ 000$

$i = 9.5\% = 0.095$

$n = 65 - 18 = 47$

$$\begin{aligned}\therefore\ A &= P(1+i)^n \\ &= 10\ 000(1+0.095)^{47} \\ &= 711\ 973.13\end{aligned}$$

Rick's deposit of \$10 000 will be worth \$711 973.13 by the time he retires at age 65. This is about 71 times his original investment.

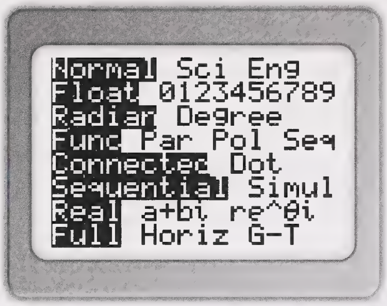
3. Textbook question 35 of "Applications and Problem Solving," p. 305

35. a. If the principal, P, is doubled, then $A = 2P$.

$$\begin{aligned}\therefore\ A &= P(1+i)^n \\ 2P &= P(1+i)^n \\ 2 &= (1+i)^n \\ \log 2 &= \log (1+i)^n \\ \log 2 &= n \log (1+i) \\ n \log (1+i) &= \log 2 \\ n &= \frac{\log 2}{\log (1+i)}\end{aligned}$$

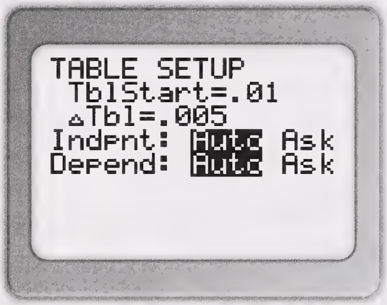
Section 1: Activity 3 (continued)

b. Interest rates are not integers; so, make sure the calculator is in Func mode.



```
Normal Sci Eng
Float 0123456789
Radian Degree
Func Par Pol Seq
Connected Dot
Sequential Simul
Real a+bi re^θi
Full Horiz G-T
```

Press 2nd [TBLSET]. Start the table at 0.01 (1%) and have the interest rates go up by increments of 0.005 (0.5%).

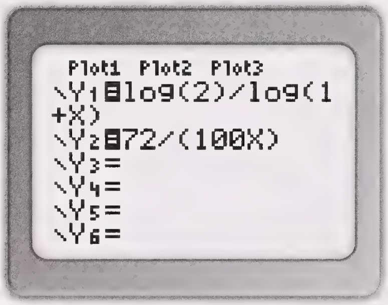


```
TABLE SETUP
 TblStart=.01
 ΔTbl=.005
Indpnt: Auto Ask
Depend: Auto Ask
```

Enter the following keystrokes:

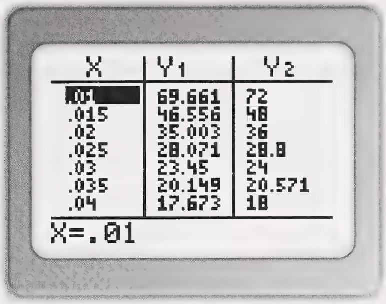
Y= LOG 2) ÷ LOG 1 + X,T,Θ,*n*) ENTER

7 2 ÷ (1 0 0 X,T,Θ,*n*)



```
 Plot1 Plot2 Plot3
\Y1■log(2)/log(1
+X)
\Y2■72/(100X)
\Y3=
\Y4=
\Y5=
\Y6=
```

Press 2nd [TABLE] to display the table.

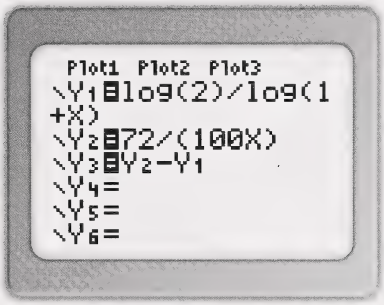


```
  X    | Y1     | Y2
.01    | 69.661 | 72
.015   | 46.556 | 48
.02    | 35.003 | 36
.025   | 28.071 | 28.8
.03    | 23.45  | 24
.035   | 20.149 | 20.571
.04    | 17.673 | 18
X=.01
```

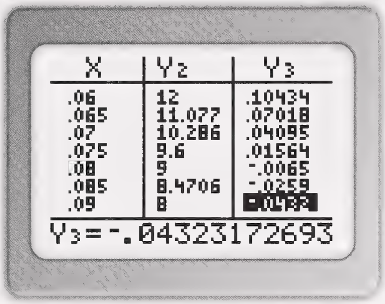
To compare the values of $\frac{\log 2}{\log(1+i)}$ and $\frac{72}{100i}$, determine the difference between the two values.

Define $Y_3 = Y_2 - Y_1$.

Y= VARS [Select the Y-VARS menu.] 1 (1: Function) 2 − VARS [Select the Y-VARS menu.] 1 (1: Function) 1 2nd [TABLE]



```
Plot1 Plot2 Plot3
\Y1=log(2)/log(1
+X)
\Y2=72/(100X)
\Y3=Y2-Y1
\Y4=
\Y5=
\Y6=
```



```
  X    | Y2     | Y3
.06    | 12     | .10434
.065   | 11.077 | .07018
.07    | 10.286 | .04095
.075   | 9.6    | .01564
.08    | 9      | -.0065
.085   | 8.4706 | -.0259
.09    | 8      | [illegible]
Y3=-.04323172693
```

If $y_3 > 0$, "rule of 72" overestimates the time required to double the principal. If $y_3 < 0$, "rule of 72" underestimates the time required to double the principal. The "rule of 72" overestimates the time required for the principal to double if the interest rate is less than 8%. Likewise, it underestimates the time required for the principal to double if the interest rate is 8% or greater. The best estimates are when the interest rates are close to 8%.

Section 1: Follow-up Activities

Extra Help

Textbook questions 7, 9.a., 18, 20, 22, 24, 26, and 28 of "Review," p. 324

7.
$$\begin{aligned} t_n &= 2(n-1) \\ t_1 &= 2(1-1) \\ &= 2(0) \\ &= 0 \end{aligned}
\qquad
\begin{aligned} t_n &= 2(n-1) \\ t_2 &= 2(2-1) \\ &= 2(1) \\ &= 2 \end{aligned}
\qquad
\begin{aligned} t_n &= 2(n-1) \\ t_3 &= 2(3-1) \\ &= 2(2) \\ &= 4 \end{aligned}
\qquad
\begin{aligned} t_n &= 2(n-1) \\ t_4 &= 2(4-1) \\ &= 2(3) \\ &= 6 \end{aligned}
\qquad
\begin{aligned} t_n &= 2(n-1) \\ t_5 &= 2(5-1) \\ &= 2(4) \\ &= 8 \end{aligned}$$

The first five terms are 0, 2, 4, 6, and 8.

9. **a.** Starting at 35, the first three multiples of 7 are 35, 42, and 49.

Notice that 7 is added to the preceding term.

$\therefore\ t_n = t_{n-1} + 7$, where $t_1 = 35$

18.
$$\begin{aligned} a &= 100 \\ r &= \frac{t_2}{t_1} \\ &= \frac{10}{100} \\ &= 0.1 \end{aligned}
\qquad
\begin{aligned} \therefore\ t_n &= ar^{n-1} \\ &= 100(0.1)^{n-1} \\ t_9 &= 100(0.1)^{9-1} \\ &= 100(0.1)^8 \\ &= 0.000\ 001 \end{aligned}$$

20.
$$\begin{aligned} a &= 600 \\ r &= \frac{t_2}{t_1} \\ &= \frac{300}{600} \\ &= 0.5 \end{aligned}
\qquad
\begin{aligned} \therefore\ t_n &= ar^{n-1} \\ &= 600(0.5)^{n-1} \\ t_8 &= 600(0.5)^{8-1} \\ &= 600(0.5)^7 \\ &= 4.6875 \end{aligned}$$

22. $a = 1$

$$r = \frac{t_2}{t_1}$$

$$= \frac{\frac{1}{3}}{1}$$

$$= \frac{1}{3}$$

$$\therefore\ t_n = ar^{n-1}$$

$$= 1\left(\frac{1}{3}\right)^{n-1}$$

$$t_6 = \left(\frac{1}{3}\right)^{6-1}$$

$$= \left(\frac{1}{3}\right)^{5}$$

$$= \frac{1}{243}$$

24. $t_4 = 5$ and $t_8 = 405$

$$\therefore\ t_n = ar^{n-1}$$

$$t_4 = ar^{4-1}$$

$$5 = ar^3 \quad ①$$

$$\therefore t_n = ar^{n-1}$$

$$t_8 = ar^{8-1}$$

$$405 = ar^7 \quad ②$$

Divide ② by ①.

$$\frac{405}{5} = \frac{ar^7}{ar^3}$$

$$81 = r^4$$

$$r^4 = 81$$

$$r = \pm\sqrt[4]{81}$$

$$= \pm 3$$

Substitute $r = 3$ into ①.

$$5 = ar^3$$

$$5 = a(3)^3$$

$$5 = 27a$$

$$a = \frac{5}{27}$$

Substitute $r = -3$ into ①.

$$5 = ar^3$$

$$5 = a(-3)^3$$

$$5 = -27a$$

$$a = -\frac{5}{27}$$

Since $t_n = ar^{n-1}$, $t_2 = ar$.

Extra Help (continued)

When $a=\frac{5}{27}$ and $r=3$,

$$\begin{aligned} t_2 &= ar \\ &= \left(\frac{5}{27}\right)(3) \\ &= \frac{5}{9} \end{aligned}$$

When $a=-\frac{5}{27}$ and $r=-3$,

$$\begin{aligned} t_2 &= ar \\ &= \left(-\frac{5}{27}\right)(-3) \\ &= \frac{5}{9} \end{aligned}$$

$\therefore t_2 = \frac{5}{9}$

26. $t_3 = 27$ and $t_7 = 2187$

$$\begin{aligned} \therefore t_n &= ar^{n-1} \\ t_3 &= ar^{3-1} \\ 27 &= ar^2 \quad ① \end{aligned}$$

$$\begin{aligned} \therefore t_n &= ar^{n-1} \\ t_7 &= ar^{7-1} \\ 2187 &= ar^6 \quad ② \end{aligned}$$

Divide ② by ①.

$$\begin{aligned} \frac{2187}{27} &= \frac{ar^6}{ar^2} \\ 81 &= r^4 \\ r^4 &= 81 \\ r &= \pm\sqrt[4]{81} \\ &= \pm 3 \end{aligned}$$

Substitute $r=3$ into ①.

$$\begin{aligned} 27 &= ar^2 \\ 27 &= a(3)^2 \\ 27 &= 9a \\ 9a &= 27 \\ a &= 3 \end{aligned}$$

Substitute $r=-3$ into ①.

$$\begin{aligned} 27 &= a(-3)^2 \\ 27 &= 9a \\ 9a &= 27 \\ a &= 3 \end{aligned}$$

Since $t_n = ar^{n-1}$, $t_{11} = ar^{10}$.

When $a = 3$ and $r = 3$,

$$t_{11} = ar^{10} = 3(3)^{10} = 177\,147$$

When $a = 3$ and $r = -3$,

$$t_{11} = ar^{10} = 3(-3)^{10} = 177\,147$$

$\therefore t_{11} = 177\,147$

28. **a.** Calculate the interest per compounding period.

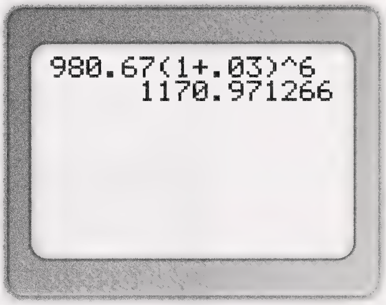
$$i = \frac{6\%}{2} = 0.03$$

Calculate the number of compounding periods.

$$n = 2 \times 3 = 6$$

$\therefore P = 980.67$, $i = 0.03$, and $n = 6$

$$\therefore A = P(1+i)^n = 980.67(1+0.03)^6 = 1170.97$$



```
980.67(1+.03)^6
          1170.971266
```

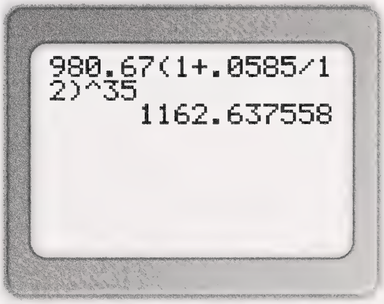
The first investment will be worth $1170.97 after three years.

b. $P = 980.67$

$$i = \frac{5.85\%}{12} = \frac{0.0585}{12}$$

$n = 35$

$$\therefore A = P(1+i)^n = 980.67\left(1+\frac{0.0585}{12}\right)^{35} = 1162.64$$



```
980.67(1+.0585/1
2)^35
          1162.637558
```

The second investment will be worth $1162.64 after 35 months.

Extra Help (continued)

c. $$\begin{aligned}\text{Total} &= 1170.97 + 1162.64\\ &= 2333.61\end{aligned}$$

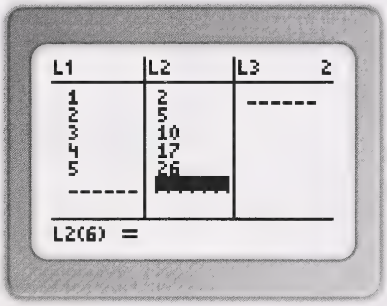
The two investments are worth $2333.61.

Enrichment

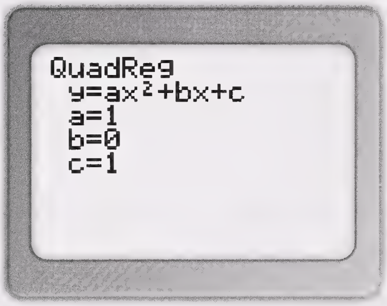
1. Set up a table of values and determine the pattern for the differences between successive terms.

n	1	2	3	4	5
t_n	2	5	10	17	26

The general term is quadratic. Enter the sequence as lists and use the Quadratic Regression feature.



```
L1     |L2    |L3      2
1      |2     |------
2      |5     |
3      |10    |
4      |17    |
5      |26    |
------ |------|
L2(6) =
```



```
QuadReg
 y=ax²+bx+c
 a=1
 b=0
 c=1
```

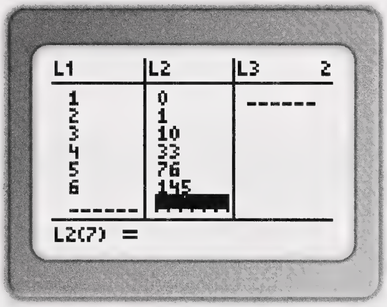
$$\begin{aligned}\therefore\ t_n &= 1n^2 + 0n + 1\\ &= n^2 + 1\end{aligned}$$

The general term is $t_n = n^2 + 1$.

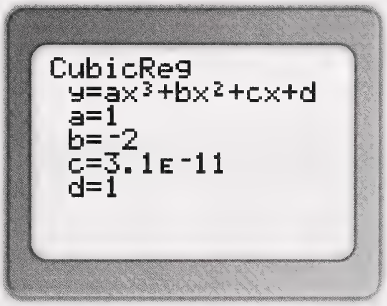
2. Set up a table of values and determine the pattern for the differences between successive terms.

n	1	2	3	4	5	6
t_n	0	1	10	33	76	145

Because $\Delta^3 t_n$ is constant, the general term is cubic. Enter the sequence as lists and use the Cubic Regression feature.



```
L1      L2      L3     2
1       0       ------
2       1
3       10
4       33
5       76
6       145
------
L2(7) =
```



```
CubicReg
 y=ax³+bx²+cx+d
 a=1
 b=-2
 c=3.1E-11
 d=1
```

Treat $c = 3.1 \times 10^{-11}$ as 0.

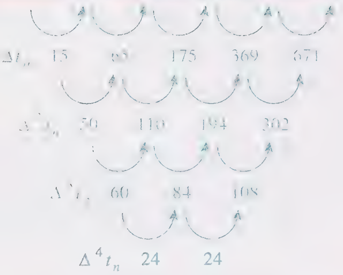
$$\therefore t_n = 1n^3 - 2n^2 + 0n + 1$$
$$= n^3 - 2n^2 + 1$$

The general term is $t_n = n^3 - 2n^2 + 1$.

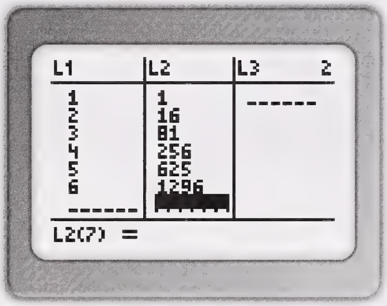
Enrichment (continued)

3. Set up a table of values and determine the pattern for the differences between successive terms.

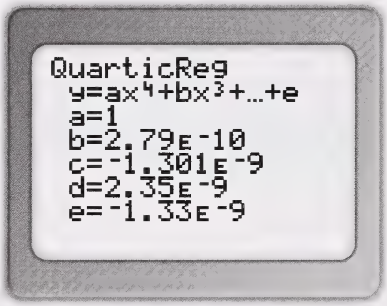
n	1	2	3	4	5	6
t_n	1	16	81	256	625	1296



Because $\Delta^4 t_n$ is constant, the general term is quartic. Enter the sequence as lists, and use the Quartic Regression feature.



```
L1      L2      L3      2
1       1       ------
2       16
3       81
4       256
5       625
6       1296
------
L2(7) =
```



```
QuarticReg
 y=ax⁴+bx³+…+e
 a=1
 b=2.79E-10
 c=-1.301E-9
 d=2.35E-9
 e=-1.33E-9
```

Here, treat b, c, d, and e as 0.

$$\therefore\ t_n = 1n^4 + 0n^3 + 0n^2 + 0n + 0$$
$$= n^4$$

The general term is $t_n = n^4$.

Section 2: Activity 1

1. a. Textbook question "Explore: Build a Formula," p. 306

Step 1: $S_n = a + ar + ar^2 + ar^3 + \ldots + ar^{n-1}$ ①

Step 2: $rS_n = ar + ar^2 + ar^3 + \ldots + ar^{n-1} + ar^n$ ②

Step 3:

$$\begin{array}{rclr} S_n & = a + & ar + ar^2 + ar^3 + \ldots + ar^{n-1} & ① \\ rS_n & = & ar + ar^2 + ar^3 + \ldots + ar^{n-1} + ar^n & ② \\ \hline S_n - rS_n & = a & - ar^n & ① - ② \end{array}$$

Step 4:

$$S_n - rS_n = a - ar^n$$

$$S_n(1-r) = a(1-r^n)$$

$$S_n = \frac{a(1-r^n)}{1-r}$$

b. Textbook questions 1 to 4 of "Inquire," p. 306

1. Equation 1, $S_n = a + ar + ar^2 + \ldots + ar^{n-1}$, is the sum of a geometric sequence. S_n represents the sum of n terms.

2. Multiplying by r in Step 2 was a helpful step because once the new equation was subtracted from the first, the result was in terms of the first term and the general term (last term). Multiplying by r enabled you to get rid of all the middle terms.

3. If $r = 1$, the expression $S_n = \dfrac{a(1-r^n)}{1-r}$ would be invalid because you cannot divide by zero.

Section 2: Activity 1 (continued)

4. **Method 1**

Use a calculator or pencil and paper.

$$1+2+4+8+16=31$$

Method 2

The sequence 1, 2, 4, 8, 16 is a geometric sequence.

$a=1$, $r=\frac{2}{1}=2$, and $n=5$

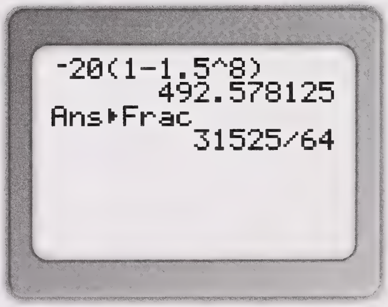
$$\begin{aligned}\therefore S_n &= \frac{a\left(1-r^n\right)}{1-r}\\ S_5 &= \frac{1\left(1-2^5\right)}{1-2}\\ &= \frac{1-32}{-1}\\ &= \frac{-31}{-1}\\ &= 31\end{aligned}$$

The formula works!

2. **Textbook questions 1 to 7, 9, 11, 13, 15, 16, and 18 of "Practice," p. 309**

1. $a=10$ and $r=\frac{15}{10}=1.5$

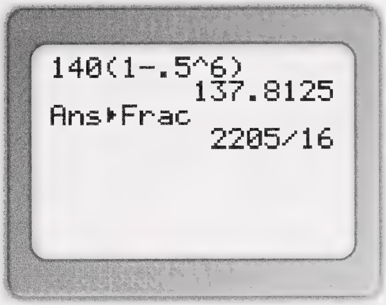
$$\begin{aligned}\therefore S_n &= \frac{a\left(1-r^n\right)}{1-r}\\ S_8 &= \frac{10\left(1-1.5^8\right)}{1-1.5}\\ &= \frac{10\left(1-1.5^8\right)}{-0.5}\\ &= -20\left(1-1.5^8\right)\\ &= 492.578\,125 \text{ or } \frac{31\,525}{64}\end{aligned}$$



```
-20(1-1.5^8)
         492.578125
Ans▸Frac
           31525/64
```

2. $a = 70$ and $r = \frac{35}{70} = 0.5$

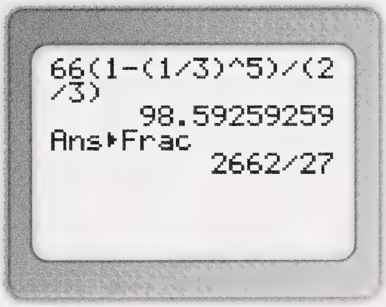
$$\begin{aligned}\therefore\ S_n &= \frac{a\left(1-r^n\right)}{1-r}\\ S_6 &= \frac{70\left(1-0.5^6\right)}{1-0.5}\\ &= \frac{70\left(1-0.5^6\right)}{0.5}\\ &= 140\left(1-0.5^6\right)\\ &= 137.8125 \text{ or } \frac{2205}{16}\end{aligned}$$



```
140(1-.5^6)
          137.8125
Ans▸Frac
           2205/16
```

3. $a = 66$ and $r = \frac{22}{66} = \frac{1}{3}$

$$\begin{aligned}\therefore\ S_n &= \frac{a\left(1-r^n\right)}{1-r}\\ S_5 &= \frac{66\left[1-\left(\frac{1}{3}\right)^5\right]}{1-\frac{1}{3}}\\ &= \frac{66\left[1-\left(\frac{1}{3}\right)^5\right]}{\frac{2}{3}}\\ &= 98.\overline{592} \text{ or } \frac{2262}{27}\end{aligned}$$

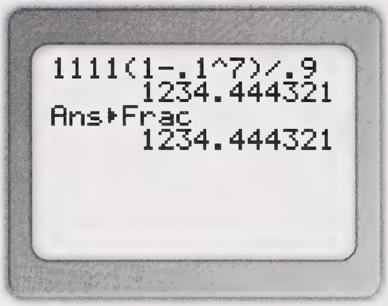


```
66(1-(1/3)^5)/(2
/3)
       98.59259259
Ans▸Frac
           2662/27
```

Section 2: Activity 1 (continued)

4. $a = 1111$ and $r = \frac{111.1}{1111} = 0.1$

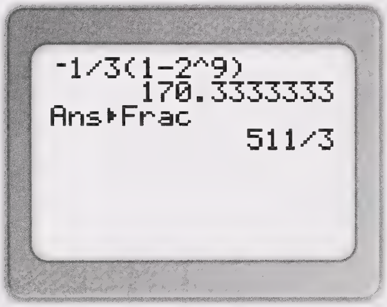
$$\begin{aligned} \therefore S_n &= \frac{a(1-r^n)}{1-r} \\ S_7 &= \frac{1111(1-0.1^7)}{1-0.1} \\ &= \frac{1111(1-0.1^7)}{0.9} \\ &\doteq 1234.4 \end{aligned}$$



```
1111(1-.1^7)/.9
          1234.444321
Ans▸Frac
          1234.444321
```

5. $a = \frac{1}{3}$, and $r = \frac{\frac{2}{3}}{\frac{1}{3}} = 2$

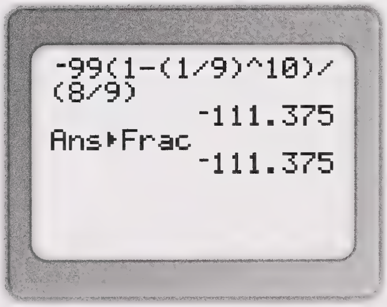
$$\begin{aligned} \therefore S_n &= \frac{a(1-r^n)}{1-r} \\ S_9 &= \frac{\frac{1}{3}(1-2^9)}{1-2} \\ &= -\frac{1}{3}(1-2^9) \\ &= 170.\bar{3} \text{ or } \frac{511}{3} \end{aligned}$$



```
-1/3(1-2^9)
          170.3333333
Ans▸Frac
                511/3
```

6. $a = -99$ and $r = \frac{-11}{-99} = \frac{1}{9}$

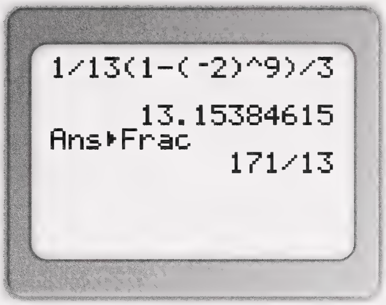
$$\begin{aligned} \therefore S_n &= \frac{a(1-r^n)}{(1-r)} \\ S_{10} &= \frac{-99\left[1-\left(\frac{1}{9}\right)^{10}\right]}{1-\frac{1}{9}} \\ &= \frac{-99\left[1-\left(\frac{1}{9}\right)^{10}\right]}{\frac{8}{9}} \\ &= -111.375 \end{aligned}$$



```
-99(1-(1/9)^10)/
(8/9)
             -111.375
Ans▸Frac
             -111.375
```

7. $a=\frac{1}{13}$ and $r=\frac{-\frac{2}{13}}{\frac{1}{13}}=-2$

$$\begin{aligned}\therefore S_n &= \frac{a(1-r^n)}{1-r}\\ S_9 &= \frac{\frac{1}{13}\left[1-(-2)^9\right]}{\left[1-(-2)\right]}\\ &= \frac{\frac{1}{13}\left[1-(-2)^9\right]}{3}\\ &= \frac{171}{13}\end{aligned}$$



```
1/13(1-(⁻2)^9)/3
         13.15384615
Ans▸Frac
              171/13
```

9. $\sum_{i=1}^{4} 10(0.1)^i = 10(0.1)^1 + 10(0.1)^2 + 10(0.1)^3 + 10(0.1)^4$

$$\begin{aligned}\therefore a &= 10(0.1)\\ r &= 0.1\\ n &= 4\end{aligned}$$

$$\begin{aligned}\therefore S_n &= \frac{a(1-r^n)}{1-r}\\ S_4 &= \frac{10(0.1)(1-0.1^4)}{1-0.1}\\ &= \frac{(1-0.1^4)}{0.9}\\ &= 1.111 \text{ or } \frac{1111}{1000}\end{aligned}$$

11. $\sum_{m=3}^{8} 200\left(\frac{1}{2}\right)^m = 200\left(\frac{1}{2}\right)^3 + 200\left(\frac{1}{2}\right)^4 + \ldots + 200\left(\frac{1}{2}\right)^8$

$$\begin{aligned}\therefore a &= 200\left(\frac{1}{2}\right)^3\\ r &= \frac{1}{2}\\ n &= 8-3+1=6\end{aligned}$$

$$\begin{aligned}\therefore S_n &= \frac{a(1-r^n)}{1-r}\\ S_6 &= \frac{200\left(\frac{1}{2}\right)^3\left[1-\left(\frac{1}{2}\right)^6\right]}{1-\frac{1}{2}}\\ &= \frac{200\left(\frac{1}{2}\right)^3\left[1-\left(\frac{1}{2}\right)^6\right]}{\frac{1}{2}}\\ &= \frac{1575}{32}\end{aligned}$$

Section 2: Activity 1 (continued)

13. $$\sum_{k=-1}^{7} 3(5)^{k-1} = 3(5)^{-1-1} + 3(5)^{0-1} + 3(5)^{1-1} + \ldots + 3(5)^{7-1}$$

$$= 3(5)^{-2} + 3(5)^{-1} + 3(5)^{0} + \ldots + 3(5)^{6}$$

$$\begin{aligned} \therefore\ a &= 3(5)^{-2} \\ r &= 5 \\ n &= 7-(-1)+1 = 9 \end{aligned}$$

$$\begin{aligned} \therefore\ S_n &= \frac{a(1-r^n)}{1-r} \\ S_9 &= \frac{3(5)^{-2}(1-5^9)}{1-5} \\ &= \frac{3(5)^{-2}(1-5^9)}{-4} \\ &= 58\ 593.72 \end{aligned}$$

15. $$\sum_{k=3}^{10} 0.3(0.15)^{k-1} = 0.3(0.15)^{3-1} + 0.3(0.15)^{4-1} + \ldots + 0.3(0.15)^{10-1}$$

$$= 0.3(0.15)^{2} + 0.3(0.15)^{3} + \ldots + 0.3(0.15)^{9}$$

$$\begin{aligned} \therefore\ a &= 0.3(0.15)^{2} \\ r &= 0.15 \\ n &= 10-3+1 = 8 \end{aligned}$$

$$\begin{aligned} \therefore\ S_n &= \frac{a(1-r^n)}{1-r} \\ S_8 &= \frac{0.3(0.15)^2(1-0.15^8)}{1-0.15} \\ &= \frac{0.3(0.15)^2(1-0.15^8)}{0.85} \\ &\doteq 0.0079 \end{aligned}$$

16. First, find the number of terms, n.

$$a = 12$$

$$r = \frac{3}{12} = \frac{1}{4}$$

$$t_n = \frac{3}{64}$$

$$\therefore\ t_n = ar^{n-1}$$

$$\frac{3}{64} = 12\left(\frac{1}{4}\right)^{n-1}$$

$$\frac{\overset{1}{\cancel{3}}}{64\times\underset{4}{\cancel{12}}} = \left(\frac{1}{4}\right)^{n-1}$$

$$\frac{1}{256} = \left(\frac{1}{4}\right)^{n-1}$$

$$\left(\frac{1}{4}\right)^4 = \left(\frac{1}{4}\right)^{n-1}$$

$$4 = n-1$$

$$n-1 = 4$$

$$n = 5$$

$$\therefore\ \sum_{k=1}^{5} 12\left(\frac{1}{4}\right)^{k-1}$$

Now, find the sum.

$$S_n = \frac{a\left(1-r^n\right)}{1-r}$$

$$S_4 = \frac{12\left[1-\left(\frac{1}{4}\right)^5\right]}{1-\frac{1}{4}}$$

$$= \frac{12\left[1-\left(\frac{1}{4}\right)^5\right]}{\frac{3}{4}}$$

$$= \frac{1023}{64} \text{ or } 15\frac{63}{64}$$

Section 2: Activity 1 (continued)

18. Find the number of terms, *n*.

$$a = -12$$

$$r = \frac{-6}{-12} = \frac{1}{2}$$

$$t_n = -\frac{3}{16}$$

$$\therefore\ t_n = ar^{n-1}$$

$$-\frac{3}{16} = -12\left(\frac{1}{2}\right)^{n-1}$$

$$\frac{3}{16} = 12\left(\frac{1}{2}\right)^{n-1}$$

$$\frac{\overset{1}{\cancel{3}}}{16 \times \underset{4}{\cancel{12}}} = \left(\frac{1}{2}\right)^{n-1}$$

$$\frac{1}{64} = \left(\frac{1}{2}\right)^{n-1}$$

$$\left(\frac{1}{2}\right)^{6} = \left(\frac{1}{2}\right)^{n-1}$$

$$6 = n - 1$$

$$n - 1 = 6$$

$$n = 7$$

$$\therefore\ \sum_{k=1}^{7} -12\left(\frac{1}{2}\right)^{k-1}$$

Now, find the sum.

$$S_n = \frac{a\left(1 - r^n\right)}{1 - r}$$

$$S_7 = \frac{-12\left[1 - \left(\frac{1}{2}\right)^7\right]}{1 - \frac{1}{2}}$$

$$= \frac{-12\left[1 - \left(\frac{1}{2}\right)^7\right]}{\frac{1}{2}}$$

$$= -\frac{381}{16} \text{ or } -23\frac{13}{16}$$

b. Textbook questions 23, 24, 25, 28, 31, and 33 of "Applications and Problem Solving," pp. 309 and 310

23. Because there are 64 entries, the first round will consist of $64 \div 2 = 32$ games. Because only the winners advance to the next round, the second round will consist of $32 \div 2 = 16$ games.

Therefore, the total number of games can be represented by the following geometric series:

$$32 + 16 + \ldots + 1$$

First, find the total number of rounds needed for the tournament.

$$a = 32$$

$$r = \frac{16}{32} = \frac{1}{2}$$

$$t_n = 1$$

$$\begin{aligned} \therefore\ t_n &= ar^{n-1} \\ 1 &= 32\left(\frac{1}{2}\right)^{n-1} \\ \frac{1}{32} &= \left(\frac{1}{2}\right)^{n-1} \\ \left(\frac{1}{2}\right)^5 &= \left(\frac{1}{2}\right)^{n-1} \\ 5 &= n-1 \\ n-1 &= 5 \\ n &= 6 \end{aligned}$$

Now, find the total number of games played.

$$\begin{aligned} S_n &= \frac{a(1-r^n)}{1-r} \\ S_6 &= \frac{32\left[1-\left(\frac{1}{2}\right)^6\right]}{1-\frac{1}{2}} \\ &= \frac{32\left[1-\left(\frac{1}{2}\right)^6\right]}{\frac{1}{2}} \\ &= 63 \end{aligned}$$

A total of 63 games are played to determine the ultimate winner.

You can think of this question in another way. Because there were 64 teams and every game eliminates one team, a total of 63 games must be played to eliminate 63 teams!

Section 2: Activity 1 (continued)

24. If 0.77% of the rain forest is lost each year, then $100\% - 0.77\% = 99.23\%$ remains.

The amounts remaining after each year form a geometric sequence.

$$775(0.9923),\ 775(0.9923)^2,\ \ldots$$

$$\begin{aligned} \therefore\ a &= 775(0.9923) \\ r &= 0.9923 \\ n &= 10 \end{aligned}$$

$$\begin{aligned} \therefore\ t_n &= ar^{n-1} \\ t_{10} &= 775(0.9923)(0.9923)^{10-1} \\ &= 775(0.9923)^{10} \\ &\doteq 717.35 \end{aligned}$$

After a decade, about 717.35 million hectares of rain forest would remain.

Therefore, $775 - 717.35 \doteq 57.65$ million hectares would be lost.

25. $100 + 100(1.10) + \ldots + 100(1.10)^{20}$

$$\begin{aligned} \therefore\ a &= 100 \\ r &= 1.10 \\ n &= 21 \end{aligned}$$

$$\begin{aligned} \therefore\ S_n &= \frac{a(1-r^n)}{1-r} \\ S_{21} &= \frac{100(1-1.10^{21})}{1-1.10} \\ &= \frac{100(1-1.10^{21})}{-0.10} \\ &= -1000(1-1.10^{21}) \\ &\doteq 6400.25 \end{aligned}$$

The aunt contributes $6400.25 to the trust fund.

28. In 3 bounces, the shock absorber crosses the equilibrium line 5 times (as shown in the diagram). Therefore, there will be a total of 6 terms in the series.

Since the shock absorber reduces the vibration by 70%, on each bounce the piston travels 30% of the distance it did on the previous bounce.

$$\begin{aligned} \text{Distance piston travels} &= 2\left[20 + 20(0.3) + 20(0.3)^2 + 20(0.3)^3 + 20(0.3)^4 + 20(0.3)^5\right] \\ &= 40 + 40(0.3) + 40(0.3)^2 + 40(0.3)^3 + 40(0.3)^4 + 40(0.3)^5 \end{aligned}$$

$$\therefore\ a = 40$$
$$r = 0.3$$
$$n = 6$$

$$\begin{aligned}\therefore\ S_n &= \frac{a(1-r^n)}{1-r}\\ S_6 &= \frac{40(1-0.3^6)}{1-0.3}\\ &= \frac{40(1-0.3^6)}{0.7}\\ &\doteq 57.10\end{aligned}$$

The total distance the piston travels is approximately 57.10 cm.

31. Let x be the annual fixed payment.

There are a total of 11 payments made from January 1, 2000, to January 1, 2010. Therefore, $n = 11$.

The total amount of money in the fund in 2010 can be represented by the following geometric series:

$$\overset{\text{Jan. 1, 2000 payment}}{x(1.10)^{10}} + \underset{\text{Jan. 1, 2001 payment}}{x(1.10)^{9}} + \ldots + x(1.10) + \overset{\text{Jan. 1, 2010 payment}}{x}$$

Write the geometric series in reverse order.

$$x + x(1.10) + \ldots + x(1.10)^9 + x(1.10)^{10}$$

$$\therefore\ a = x$$
$$r = 1.10$$
$$S_{11} = 100\ 000$$

$$\begin{aligned}\therefore\ S_n &= \frac{a(1-r^n)}{1-r}\\ S_{11} &= \frac{x(1-1.10^{11})}{1-1.10}\\ 100\ 000 &= \frac{x(1-1.10^{11})}{-0.10}\\ -10\ 000 &= x(1-1.10^{11})\\ x &= \frac{-10\ 000}{(1-1.10^{11})}\\ &= 5396.31\end{aligned}$$

Each individual payment is \$5396.31.

Section 2: Activity 1 (continued)

33. $\displaystyle\sum_{k=1}^{10} t_k = 244\sum_{k=1}^{5} t_k$

$$
\begin{aligned}
\therefore\quad \frac{a\left(1-r^{n}\right)}{1-r} &= 244\times\frac{a\left(1-r^{n}\right)}{1-r}\\
\frac{\cancel{t_1}\left(1-r^{10}\right)}{\cancel{1-r}} &= 244\times\frac{\cancel{t_1}\left(1-r^{5}\right)}{\cancel{1-r}}\\
1-r^{10} &= 244\left(1-r^{5}\right)\\
\frac{1-r^{10}}{1-r^{5}} &= 244\\
\frac{-1\left(r^{10}-1\right)}{-1\left(r^{5}-1\right)} &= 244\\
\frac{\left(r^{10}-1\right)}{\left(r^{5}-1\right)} &= 244\\
\frac{\cancel{\left(r^{5}-1\right)}\left(r^{5}+1\right)}{\cancel{\left(r^{5}-1\right)}} &= 244\\
r^{5}+1 &= 244\\
r^{5} &= 243\\
r &= \sqrt[5]{243}\\
&= 3
\end{aligned}
$$

For the geometric sequence $t_1, t_2, t_3, \ldots,$ the first term, a, is t_1.

The common ratio for the sequence is 3.

3. Determine the interest per compounding period.

$$
\begin{aligned}
i &= \frac{6\%}{12}\\
&= 0.005
\end{aligned}
$$

The following geometric series represents the amount of money invested by the Johnstons:

$$\underset{\uparrow\text{— first month}}{100(1.005)^{12}} + 100(1.005)^{11} + \ldots + 100(1.005)^{2} + \underset{\uparrow\text{— last month}}{100(1.005)}$$

Find the sum.

$$a = 100(1.005)^{12}$$
$$r = 1.005^{-1}$$
$$n = 12$$

$$\therefore\ S_n = \frac{a\left(1 = r^n\right)}{1-r}$$

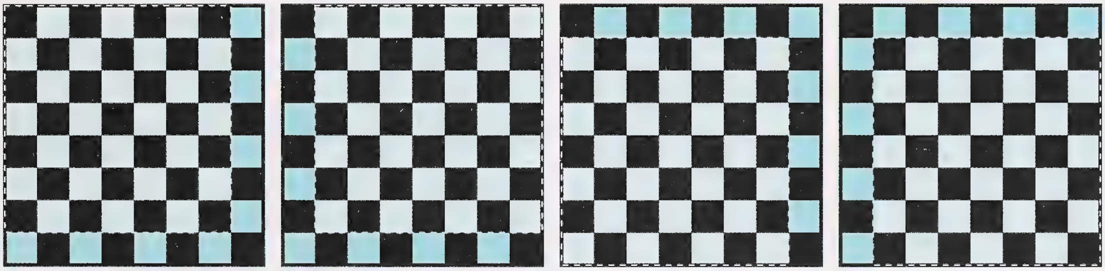
$$S_{12} = \frac{100(1.005)^{12}\left[1-\left(1.005^{-1}\right)^{12}\right]}{1-1.005^{-1}}$$
$$= \frac{100(1.005)^{12}\left(1-1.005^{-12}\right)}{1-1.005^{-1}}$$
$$= 1239.72$$

The Johnstons will have $239.72 more invested in RRSPs than the Hays.

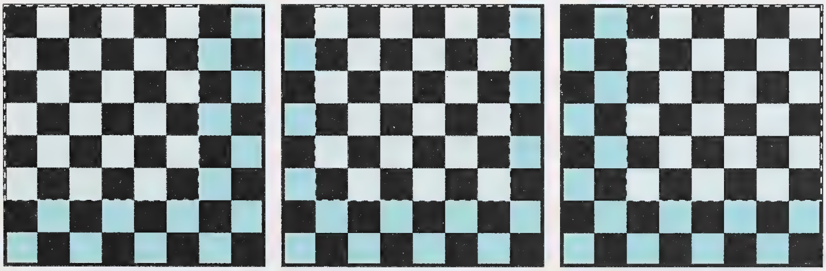
4. Work with the largest squares first.

There is one 8 × 8 square, namely the checker board itself.

There are four 7 × 7 squares, one in each of the four corners.



There are nine 6 × 6 squares: three across the top, three across the centre of the board, and three across the bottom. The three across the top are illustrated.



Notice the pattern.

$$1 + 4 + 9 + \ldots$$

Section 2: Activity 1 (continued)

These are the perfect squares. Therefore, the total number of squares is represented by the following series:

$$\underset{\substack{\uparrow\\ 8\times8\\ \text{square}}}{1} + \underset{\substack{\uparrow\\ 7\times7\\ \text{squares}}}{4} + \underset{\substack{\uparrow\\ 6\times6\\ \text{squares}}}{9} + \underset{\substack{\uparrow\\ 5\times5\\ \text{squares}}}{16} + \underset{\substack{\uparrow\\ 4\times4\\ \text{squares}}}{25} + \underset{\substack{\uparrow\\ 3\times3\\ \text{squares}}}{36} + \underset{\substack{\uparrow\\ 2\times2\\ \text{squares}}}{49} + \underset{\substack{\uparrow\\ 1\times1\\ \text{squares}}}{64}$$

There are 204 squares on a checkerboard.

Section 2: Activity 2

1. a. **Textbook question "Explore: Use a Pattern," p. 313**

The sequence of the distances moved is as follows:

$$50\text{ m}, 25\text{ m}, \frac{25}{2}\text{ m}, \frac{25}{4}\text{ m}, \frac{25}{8}\text{ m}, \ldots$$

b. **Textbook questions 1 to 4 of "Inquire," p. 313**

1. The sequence is geometric because there is a common ratio of $\frac{1}{2}$.

2. $$\begin{aligned} 50+25+\frac{25}{2} &= 50+25+12\frac{1}{2} \\ &= 87\frac{1}{2} \end{aligned}$$

After three steps, you moved $87\frac{1}{2}$ m towards the doorway.

$$\begin{aligned} 50+25+\frac{25}{2}+\frac{25}{4}+\frac{25}{8} &= 50+25+12\frac{1}{2}+6\frac{1}{4}+3\frac{1}{8} \\ &= 50+25+12\frac{4}{8}+6\frac{2}{8}+3\frac{1}{8} \\ &= 96\frac{7}{8} \end{aligned}$$

After five steps, you moved $96\frac{7}{8}$ m towards the doorway.

You will never reach the doorway because the process can be continued indefinitely.

3. The expression $S_n = \frac{a(1-r^n)}{1-r}$, where $a = 50$, $r = \frac{1}{2}$, and $n = 100$, could be used to calculate the total distance moved.

   If you moved an infinite number of steps, the total you would have moved is 100 m.

4. The expression $S_n = \frac{a(1-r^n)}{1-r}$ simplifies to $S_n = \frac{a}{1-r}$ for $-1 < r < 1$ because r^n becomes so small and so close to zero that you assume it is zero.

   $$\begin{aligned} \therefore\ S_n &= \frac{a(1-0)}{1-r} \\ &= \frac{a}{1-r} \end{aligned}$$

   The reason r^n gets close to zero is because the ratio will be a fraction where the denominator is greater than the numerator. As n gets larger, the power of the fraction gets smaller.

   If $r > 1$, r^n will increase as n increases. If $r < -1$, r^n will alternate between positive and negative numbers of increasing magnitude.

   Therefore, if $r > 1$ or $r < -1$, then $S_n \neq \frac{a}{1-r}$.

2. Textbook questions 1, 4, 5, 6, 7, 9, 14, and 15 of "Practice," p. 316

1. $a = 10$ and $r = \dfrac{1}{10} = 0.1$

   Since $r = 0.1$, the sum exists.

   $$\begin{aligned} \therefore\ S &= \frac{a}{1-r} \\ &= \frac{10}{1-0.1} \\ &= \frac{10}{0.9} \\ &\doteq \frac{100}{9} \end{aligned}$$

Section 2: Activity 2 (continued)

4. $a = 625$ and $r = \frac{25}{625} = 0.04$

Since $r = 0.04$, the sum exists.

$$\begin{aligned} \therefore\ S &= \frac{a}{1-r} \\ &= \frac{625}{1-0.04} \\ &= \frac{625}{0.96} \\ &= \frac{15\,625}{24} \end{aligned}$$

5. $a = -81$ and $r = \frac{-27}{-81} = \frac{1}{3}$

Since $r = \frac{1}{3}$, the sum exists.

$$\begin{aligned} \therefore\ S &= \frac{a}{1-r} \\ &= \frac{-81}{1-\frac{1}{3}} \\ &= \frac{-81}{\frac{2}{3}} \\ &= \frac{-243}{2} \text{ or } -121\frac{1}{2} \end{aligned}$$

6. $a = 1$ and $r = \frac{x}{1} = x$

The sum of the series can only be determined if the common ratio lies between 1 and -1.

$$\begin{aligned} \therefore\ S &= \frac{a}{1-r},\ \text{where } -1 < r < 1 \\ &= \frac{1}{1-x},\ \text{where } -1 < x < 1 \end{aligned}$$

If $x \geq 1$ or $x \leq -1$, the sum cannot be determined.

7. $a = 1$ and $r = \frac{-1}{1} = -1$

The sum cannot be determined.

In this case, if the number of terms is even, then the sum is 0; if the number of terms is odd, then the sum is 1.

9.
$$\begin{aligned}\sum_{i=0}^{\infty} 2(0.1)^i &= 2(0.1)^0 + 2(0.1)^1 + 2(0.1)^2 + 2(0.1)^3 + \ldots \\ &= 2(1) + 2(0.1) + 2(0.01) + 2(0.001) + \ldots \\ &= 2 + 0.2 + 0.02 + 0.002 + \ldots\end{aligned}$$

$$\begin{aligned}\therefore\ a &= 2 \\ r &= 0.1\end{aligned} \qquad \begin{aligned}\therefore\ S &= \frac{a}{1-r} \\ &= \frac{2}{1-0.1} \\ &= \frac{2}{0.9} \\ &= \frac{20}{9} \\ &= 2.\overline{2}\end{aligned}$$

14.
$$\begin{aligned}\sum_{i=1}^{\infty} 3\left(\frac{7}{6}\right)^i &= 3\left(\frac{7}{6}\right)^1 + 3\left(\frac{7}{6}\right)^2 + 3\left(\frac{7}{6}\right)^3 + 3\left(\frac{7}{6}\right)^4 + \ldots \\ &= \overset{1}{\cancel{3}}\left(\frac{7}{\underset{2}{\cancel{6}}}\right) + \overset{1}{\cancel{3}}\left(\frac{49}{\underset{12}{\cancel{36}}}\right) + \overset{1}{\cancel{3}}\left(\frac{343}{\underset{72}{\cancel{216}}}\right) + \overset{1}{\cancel{3}}\left(\frac{2401}{\underset{432}{\cancel{1296}}}\right) + \ldots \\ &= \frac{7}{2} + \frac{49}{12} + \frac{343}{72} + \frac{2401}{432} + \ldots\end{aligned}$$

Because r is greater than 1, the sum cannot be determined.

15.
$$\begin{aligned}\sum_{i=1}^{\infty} 2(1+0.1)^i = \sum_{i=1}^{\infty} 2(1.1)^i &= 2(1.1)^1 + 2(1.1)^2 + 2(1.1)^3 + 2(1.1)^4 + \ldots \\ &= 2.2 + 2.42 + 2.662 + 2.9282 + \ldots\end{aligned}$$

Because r is greater than 1, the sum cannot be determined.

Section 2: Activity 2 (continued)

3. Textbook questions 17.c., 17.d., 18, 19, and 22 of "Applications and Problem Solving," pp. 316 and 317

17. c. $0.\overline{21} = 0.21 + 0.0021 + 0.000\,021 + \ldots$

$$\begin{aligned} \therefore\ a &= 0.21 = \frac{21}{100} \\ r &= \frac{1}{100} \end{aligned}$$

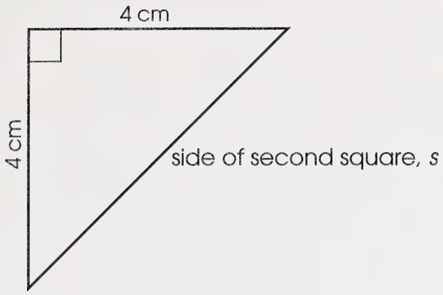
$$\begin{aligned} \therefore\ S &= \frac{a}{1-r} \\ &= \frac{\frac{21}{100}}{1-\frac{1}{100}} \\ &= \frac{\frac{21}{100}}{\frac{99}{100}} \\ &= \frac{21}{99} \\ &= \frac{7}{33} \end{aligned}$$

d. $0.6\overline{72} = 0.6 + 0.072 + 0.000\,72$

$$\begin{aligned} \therefore\ a &= 0.072 = \frac{72}{1000} \\ r &= \frac{1}{100} \end{aligned}$$

$$\begin{aligned} \therefore\ S &= 0.6 + \frac{a}{1-r} \\ &= \frac{6}{10} + \frac{\frac{72}{1000}}{1-\frac{1}{100}} \\ &= \frac{6}{10} + \frac{\frac{72}{1000}}{\frac{99}{100}} \\ &= \frac{6}{10} + \frac{72}{990} \\ &= \frac{66}{110} + \frac{8}{110} \\ &= \frac{74}{110} \\ &= \frac{37}{55} \end{aligned}$$

18. a.



$$s^2 = 4^2 + 4^2$$
$$s^2 = 32$$
$$s = 4\sqrt{2}$$

The side of the second square is $4\sqrt{2}$ cm.

b. Find the perimeter of the first square.

$$\begin{aligned} \text{Perimeter} &= 4s \\ &= 4(8) \\ &= 32 \text{ cm} \end{aligned}$$

Find the perimeter of the second square.

$$\begin{aligned} \text{Perimeter} &= 4s \\ &= 4(4\sqrt{2}) \\ &= 16\sqrt{2} \text{ cm} \end{aligned}$$

Section 2: Activity 2 (continued)

Find the perimeter of the third square.

$$\begin{aligned} s^2 &= (2\sqrt{2})^2 + (2\sqrt{2})^2 \\ s^2 &= 8+8 \\ s^2 &= 16 \\ s &= 4 \end{aligned}$$

$$\begin{aligned} \therefore \text{ Perimeter} &= 4s \\ &= 4(4) \\ &= 16 \text{ cm} \end{aligned}$$

The perimeters form the geometric sequence $32, 16\sqrt{2}, 16, \ldots$.

Find the sum of the perimeters.

$$a = 32 \text{ and } r = \frac{16\sqrt{2}}{32} = \frac{\sqrt{2}}{2}$$

$$\begin{aligned} \therefore\ S &= \frac{a}{1-r} \\ &= \frac{32}{1-\frac{\sqrt{2}}{2}} \\ &= \frac{32}{\frac{2-\sqrt{2}}{2}} \\ &= \frac{64}{2-\sqrt{2}} \\ &= \frac{64}{2-\sqrt{2}} \times \frac{2+\sqrt{2}}{2+\sqrt{2}} \quad \leftarrow \text{Rationaliz} \\ &= \frac{64(2+\sqrt{2})}{4-2} \\ &= \frac{\overset{32}{\cancel{64}}(2+\sqrt{2})}{\underset{1}{\cancel{2}}} \\ &= 64 + 32\sqrt{2} \end{aligned}$$

The sum of the perimeters of the infinite nest of squares is $(64+32\sqrt{2})$ cm.

c. Area of first square $= s^2$

$$= 8^2$$

$$= 64 \text{ cm}^2$$

Area of second square $= s^2$

$$= \left(4\sqrt{2}\right)^2$$

$$= 32 \text{ cm}^2$$

Area of third square $= s^2$

$$= 4^2$$

$$= 16 \text{ cm}^2$$

The areas form the geometric sequence 64, 32, 16, … .

Find the sum of the areas of the infinite nest.

$$a = 64 \text{ and } r = \frac{32}{64} = \frac{1}{2}$$

$$\begin{aligned} \therefore\ S &= \frac{a}{1-r} \\ &= \frac{64}{1-\frac{1}{2}} \\ &= \frac{64}{\frac{1}{2}} \\ &= 128 \end{aligned}$$

The sum of the areas of the infinite nest of squares is 128 cm^2.

d. The area of the first square is 64 cm^2. Since the total sum of the area is 128 cm^2, the total sum of the areas of all the other nested squares is $128 - 64 = 64 \text{ cm}^2$. This is equal to the area of the first square.

Section 2: Activity 2 (continued)

19. **a.** If the decline in lead production continues this trend, the following geometric sequence, where $r = \frac{224\ 000}{242\ 000} = \frac{112}{121}$, will represent the lead production.

$$224\ 000\left(\frac{112}{121}\right), 224\ 000\left(\frac{112}{121}\right)^2, 224\left(\frac{112}{121}\right)^3, \ldots$$

Find the total mass of lead produced from 1998 to 2001.

$$a = 224\ 000\left(\frac{112}{121}\right) \text{ and } r = \frac{112}{121}$$

$$\begin{aligned} \therefore S_n &= \frac{a(1-r^n)}{1-r} \\ S_4 &= \frac{224\ 000\left(\frac{112}{121}\right)\left[1-\left(\frac{112}{121}\right)^4\right]}{1-\frac{112}{121}} \\ &= \frac{224\ 000\left(\frac{112}{121}\right)\left[1-\left(\frac{112}{121}\right)^4\right]}{\frac{9}{121}} \\ &= \frac{224\ 000(112)\left[1-\left(\frac{112}{121}\right)^4\right]}{9} \\ &\doteq 741\ 327 \end{aligned}$$

Approximately 741 327 t of lead was produced from 1998 to 2001.

b. $a = 242\ 000$ and $r = \frac{112}{121}$

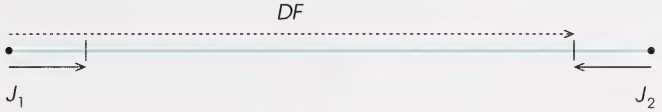
$$\begin{aligned} \therefore S &= \frac{a}{1-r} \\ &= \frac{242\ 000}{1-\frac{112}{121}} \\ &= \frac{242\ 000}{\frac{9}{121}} \\ &\doteq 3\ 253\ 556 \end{aligned}$$

Approximately 3 253 556 t of lead would be produced from 1996 on.

22. a. Let J_1 represent the first jogger; let J_2 represent the second jogger; and let DF represent the dragonfly.

When the dragonfly meets the second jogger, the combined distance travelled is 10 km.

Since $d = vt$, the dragonfly travels a distance of $45t$ and the second jogger travels a distance of $8t$.



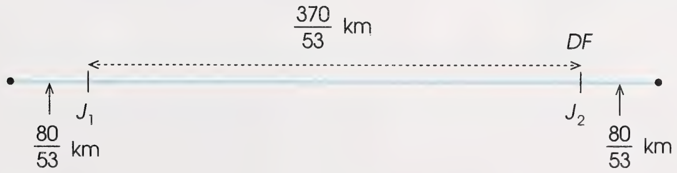
Because the dragonfly and the second jogger travel during the same period of time,

$$\begin{aligned} 45t+8t&=10\\ 53t&=10\\ t&=\frac{10}{53}\text{ h}\end{aligned}$$

Therefore, the distance DF travels to meet J_2 is $45\left(\frac{10}{53}\right)$ km.

Now, find the distance between J_1 and DF/J_2.

$$\begin{aligned} 10-(8t+8t)&=10-16t\\ &=10-16\left(\frac{10}{53}\right)\\ &=\frac{530}{53}-\frac{160}{53}\\ &=\frac{370}{53}\end{aligned}$$



Section 2: Activity 2 (continued)

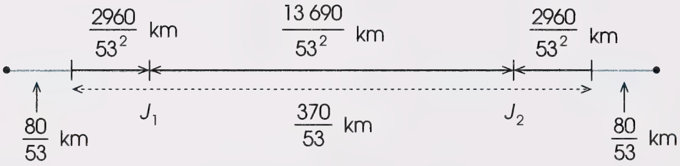
When the dragonfly meets the first jogger, the combined distance travelled is $\frac{370}{53}$ km.

$$\begin{aligned} \therefore\ 45t+8t &= \frac{370}{53} \\ 53t &= \frac{370}{53} \\ t &= \frac{370}{53^2} \end{aligned}$$

Therefore, the distance *DF* travels to meet J_1 is $45\left(\frac{370}{53^2}\right)$ km.

Now, find the distance between DF/J_1 and J_2.

$$\begin{aligned} \frac{370}{53}-2(8t) &= \frac{370}{53}-2\left(8\times\frac{370}{53^2}\right) \\ &= \frac{19\,610}{53^2}-\frac{5920}{53^2} \\ &= \frac{13\,690}{53^2} \end{aligned}$$



$$\begin{aligned} \therefore\ 45t+8t &= \frac{13\,690}{53^2} \\ 53t &= \frac{13\,690}{53^2} \\ t &= \frac{13\,690}{53^3} \end{aligned}$$

Therefore, the distance *DF* travels to meet J_2 again is $45\left(\frac{13\,690}{53^3}\right)$ km.

Thus, the geometric series that represents the sum of the distances that the dragonfly travels is

$$45\left(\frac{10}{53}\right)+45\left(\frac{370}{53^2}\right)+45\left(\frac{13\,690}{53^3}\right)+\ldots$$

b. $a = \dfrac{450}{53}$

$r = \dfrac{37}{53}$

$$\begin{aligned} \therefore\ S &= \frac{a}{1-r} \\ &= \frac{\frac{450}{53}}{1-\frac{37}{53}} \\ &= \frac{\frac{450}{\cancel{53}}}{\frac{16}{\cancel{53}}} \\ &= \frac{450}{16} \\ &= \frac{225}{8} \text{ or } 28\frac{1}{8} \end{aligned}$$

The dragonfly will fly a total distance of $28\frac{1}{8}$ km.

c. Each jogger will run 5 km before they meet. The time it will take the joggers to run 5 km is $\frac{5}{8}$ h.

Therefore, the dragonfly will fly for $\frac{5}{8}$ h as well. Find the total distance the dragonfly covers.

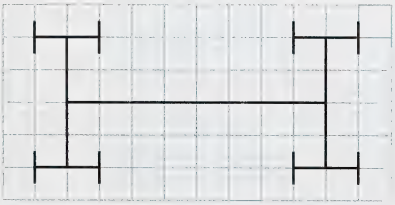
Note:
$d = vt$
$5 = 8t$
$t = \frac{5}{8}$

$$\begin{aligned} d &= 45 \times \frac{5}{8} \\ &= \frac{225}{8} \\ &= 28\frac{1}{8} \text{ or } 28.125 \end{aligned}$$

The dragonfly will fly a total distance of $28\frac{1}{8}$ km.

4. **a. Textbook questions 1 to 6 of Investigation 2, "Fractals," p. 284**

1.



Section 2: Activity 2 (continued)

2.

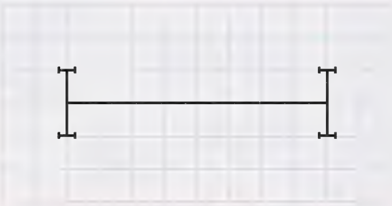
Number of steps	Total length of all sides
1	1
2	$1+\frac{1}{2}+\frac{1}{2}=2$
3	$1+\frac{1}{2}+\frac{1}{2}+\frac{1}{4}+\frac{1}{4}+\frac{1}{4}+\frac{1}{4}=3$
4	4
…	…

3. The series $1+\frac{1}{2}+\frac{1}{2}+\frac{1}{4}+\frac{1}{4}+\frac{1}{4}+\frac{1}{4}+\ldots$ can be regrouped as follows:

$$
\begin{aligned}
1+\frac{1}{2}+\frac{1}{2}+\frac{1}{4}+\frac{1}{4}+\frac{1}{4}+\frac{1}{4}+\ldots &= 1+\left(\frac{1}{2}+\frac{1}{2}\right)+\left(\frac{1}{4}+\frac{1}{4}+\frac{1}{4}+\frac{1}{4}\right)+\ldots \\
&= 1+1+1+\ldots
\end{aligned}
$$

Therefore, the sum is infinite.

4.



5.

Number of steps	Total length of all lines
1	1
2	$1+\frac{1}{4}+\frac{1}{4}=1\frac{1}{2}$
3	$1+\frac{1}{4}+\frac{1}{4}+\frac{1}{16}+\frac{1}{16}+\frac{1}{16}+\frac{1}{16}=1\frac{3}{4}$
4	$1\frac{7}{8}$
…	…

6. Consider how the total length changes from step to step.

$$1, 1\frac{1}{2}, 1\frac{3}{4}, 1\frac{7}{8}, \ldots$$

The sum approaches 2. Therefore, the total length of this fractal is 2.

b. Textbook question 24 of "Applications and Problem Solving," p. 317

24. a. The total length of the first fractal is infinite.

$$\begin{aligned} 1+\left(\frac{1}{2}+\frac{1}{2}\right)+\left(\frac{1}{4}+\frac{1}{4}+\frac{1}{4}+\frac{1}{4}\right)+\ldots &= 1+1+1+\ldots \\ &= \infty \end{aligned}$$

b. The total length of the second fractal is given by the following infinite geometric series:

$$1+\left(\frac{1}{4}+\frac{1}{4}\right)+\left(\frac{1}{16}+\frac{1}{16}+\frac{1}{16}+\frac{1}{16}\right)+\ldots = 1+\frac{1}{2}+\frac{1}{4}+\ldots$$

$$\therefore\ a = 1 \text{ and } r = \frac{1}{2}$$

$$\begin{aligned} \therefore\ S &= \frac{a}{1-r} \\ &= \frac{1}{1-\frac{1}{2}} \\ &= \frac{1}{\frac{1}{2}} \\ &= 2 \end{aligned}$$

The total length of the second fractal is 2 units.

c. Answers may vary.

The result for the first fractal is surprising. The length of this fractal is infinite, yet it can be drawn on a single sheet of graph paper.

Section 2: Follow-up Activities

Extra Help

Textbook questions 16, 17, 18, 20, and 24 of "Chapter Check," p. 326

16. $1000+500+250+\ldots+15.625$

First, find the number of terms, *n*.

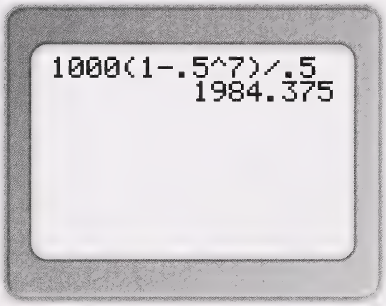
$$\begin{aligned} a &= 1000 \\ r &= \frac{500}{1000}=0.5 \\ t_n &= 15.625 \end{aligned}$$

$$\begin{aligned} \therefore\ t_n &= ar^{n-1} \\ 15.625 &= 1000(0.5)^{n-1} \\ 0.015\,625 &= 0.5^{n-1} \\ 0.5^6 &= 0.5^{n-1} \\ 6 &= n-1 \\ n &= 7 \end{aligned}$$

Use guess and check to determine that $0.015\,625=0.5^6$.

Now, determine the sum of the series.

$$\begin{aligned} S_n &= \frac{a\left(1-r^n\right)}{1-r} \\ S_7 &= \frac{1000\left(1-0.5^7\right)}{1-0.5} \\ &= \frac{1000\left(1-0.5^7\right)}{0.5} \\ &= 1984.375 \end{aligned}$$



```
1000(1-.5^7)/.5
         1984.375
```

17. $\sum\limits_{i=0}^{\infty}75(0.1)^i=75(0.1)^0+75(0.1)^1+75(0.1)^2+\ldots$

$$=75+7.5+0.75+\ldots$$

$$\begin{aligned} \therefore\ a &= 75 \\ r &= 0.1 \end{aligned}$$

$$\begin{aligned} \therefore\ S &= \frac{a}{1-r} \\ &= \frac{75}{1-0.1} \\ &= \frac{75}{0.9} \\ &= 83.\overline{3} \end{aligned}$$

18. $$\sum_{k=1}^{6} 3^{k-1} = 3^{1-1} + 3^{2-1} + 3^{3-1} + \ldots + 3^{6-1}$$

$$= 3^0 + 3^1 + 3^2 + \ldots + 3^5$$

$$= 1 + 3 + 9 + \ldots + 243$$

$$\therefore\ a = 1$$

$$r = \frac{3}{1} = 3$$

$$\therefore\ S_n = \frac{a(1-r^n)}{1-r}$$

$$S_6 = \frac{1(1-3^6)}{1-3}$$

$$= \frac{1-729}{-2}$$

$$= \frac{-728}{-2}$$

$$= 364$$

20. a. The following sequence represents the number of cells at each stage of cytokinesis.

first stage

1, 2, 4, 8, …

original cell

Find the general term of the sequence.

$$a = 1$$

$$r = \frac{2}{1} = 2$$

$$\therefore\ t_n = ar^{n-1}$$

$$= 1(2^{n-1})$$

$$= 2^{n-1}$$

Determine the number of times cytokinesis takes place within 30 minutes.

$$\frac{30 \text{ min}}{3 \text{ min}} = 10$$

Because cytokinesis takes place 10 times within 30 minutes, determine t_{11}.

$$t_n = 2^{n-1}$$

$$t_{11} = 2^{11-1}$$

$$= 2^{10}$$

$$= 1024$$

After 30 minutes, 1024 cells would be present.

Extra Help (continued)

b. Determine the number of terms, *n*.

$$\begin{aligned} t_n &= 2^{n-1} \\ 32\,768 &= 2^{n-1} \\ 2^{15} &= 2^{n-1} \\ 15 &= n-1 \\ n &= 16 \end{aligned}$$

Use guess and check to determine that $32\,768 = 2^{15}$.

Because 32 768 is the sixteenth term of the sequence, the cell has gone through 15 stages of cytokinesis. Therefore, the cell has been undergoing cytokinesis for $15 \times 3 = 45$ minutes.

24. a. $\frac{1}{6} + \frac{13}{6^2} + \frac{19}{6^3} + \frac{97}{6^4} + \ldots + \frac{3^n + (-2)^n}{6^n} + \ldots$

First, find the fifth term, t_5.

$$\begin{aligned} t_n &= \frac{3^n + (-2)^n}{6^n} \\ t_5 &= \frac{3^5 + (-2)^5}{6^5} \\ &= \frac{211}{6^5} \end{aligned}$$

Now, find the first five partial sums.

$$\begin{aligned} S_1 &= t_1 \\ &= \frac{1}{6} \end{aligned}$$

$$\begin{aligned} S_2 &= S_1 + t_2 \\ &= \frac{1}{6} + \frac{13}{6^2} \\ &= \frac{6}{6^2} + \frac{13}{6^2} \\ &= \frac{19}{6^2} \end{aligned}$$

$$\begin{aligned} S_3 &= S_2 + t_3 \\ &= \frac{19}{6^2} + \frac{19}{6^3} \\ &= \frac{114}{6^3} + \frac{19}{6^3} \\ &= \frac{133}{6^3} \end{aligned}$$

$$\begin{aligned} S_4 &= S_3 + t_4 \\ &= \frac{133}{6^3} + \frac{97}{6^4} \\ &= \frac{798}{6^4} + \frac{97}{6^4} \\ &= \frac{895}{6^4} \end{aligned}$$

$$\begin{aligned} S_5 &= S_4 + t_5 \\ &= \frac{895}{6^4} + \frac{211}{6^5} \\ &= \frac{5370}{6^5} + \frac{211}{6^5} \\ &= \frac{5581}{6^5} \end{aligned}$$

b. $\displaystyle\sum_{n=1}^{\infty}\frac{3^n+(-2)^n}{6^n}$

c. $\displaystyle\frac{1}{6}+\frac{13}{6^2}+\frac{19}{6^3}+\frac{97}{6^4}+\ldots+\frac{3^n+(-2)^n}{6^n}+\ldots$

$$=\frac{3^1+(-2)^1}{6^1}+\frac{3^2+(-2)^2}{6^2}+\frac{3^3+(-2)^3}{6^3}+\frac{3^4+(-2)^4}{6^4}+\ldots$$

$$=\left(\frac{3^1}{6^1}+\frac{3^2}{6^2}+\frac{3^3}{6^3}+\frac{3^4}{6^4}+\ldots\right)+\left(\frac{(-2)^1}{6^1}+\frac{(-2)^2}{6^2}+\frac{(-2)^3}{6^3}+\frac{(-2)^4}{6^4}+\ldots\right)$$

$$=\left(\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\ldots\right)+\left(-\frac{1}{3}+\frac{1}{9}-\frac{1}{27}+\frac{1}{81}+\ldots\right)$$

[illegible] and $r =$ [illegible] $a = -\frac{1}{3}$ and r [illegible]

$$=\frac{\frac{1}{2}}{1-\frac{1}{2}}+\frac{-\frac{1}{3}}{1-\left(-\frac{1}{3}\right)}$$

$$=\frac{\frac{1}{2}}{\frac{1}{2}}+\frac{-\frac{1}{3}}{\frac{4}{3}}$$

$$=1+\left(-\frac{1}{3}\right)\left(\frac{3}{4}\right)$$

$$=1-\frac{1}{4}$$

$$=\frac{3}{4}$$

Therefore, the likely sum is $\frac{3}{4}$.

Enrichment

1. a. **Textbook question "Explore: Use a Code," p. 318**

To decode the sentence, count the number of letters in each word.

3.141 592 653 579 793 2

b. **Textbook questions 1 to 7 of "Inquire," p. 318**

1. $\frac{22}{7} \doteq 3.142\ 857\ 143$

2. $\left(\frac{16}{9}\right)^2 \doteq 3.160\ 493\ 827$

3. $3\frac{10}{71} < \pi < 3\frac{1}{7}$

$3.140\ 845\ 070 < \pi < 3.142\ 857\ 143$

No, the approximations found in the previous two questions are not within this range. The approximation from the Rhind Papyrus is larger than $3\frac{1}{7}$; and the approximation used by the Ancient Egyptians is equal to $3\frac{1}{7}$, which is not within the range.

4. $3+\frac{8}{60}+\frac{30}{60^2} \doteq 3.141\ 666\ 667$

5. $\frac{355}{113} \doteq 3.141\ 592\ 92$

6. $\frac{3927}{1250} = 3.1416$

7. Tsu Ch'ung-chih's approximation for π is closest to the calculator's approximation for π.

2. **a. Textbook questions 1, 3, 4, and 6 of "Practice," p. 320**

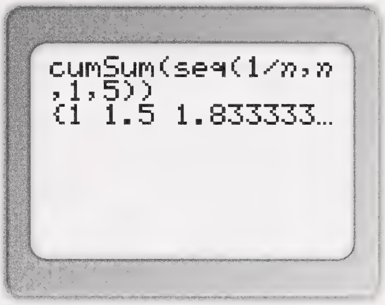
1. $\sum_{n=1}^{\infty} \frac{1}{n}$

2nd [LIST] [Select the OPS menu.] 6

2nd [LIST] [Select the OPS menu.] 5

1 ÷ X,T,Θ,*n* , X,T,Θ,*n* , 1 ,

5)) ENTER



```
cumSum(seq(1/n,n
,1,5))
{1 1.5 1.833333...
```

The first five partial sums are $1,\ 1.5,\ 1.8\overline{3},\ 2.08\overline{3}$, and $2.28\overline{3}$.

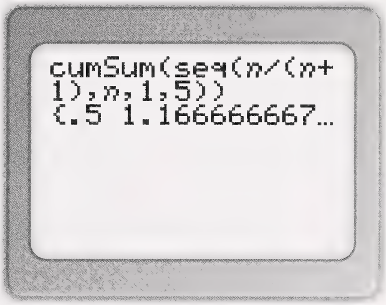
3. $\sum_{n=1}^{\infty} \frac{n}{n+1}$

2nd [LIST] [Select the OPS menu.] 6

2nd [LIST] [Select the OPS menu.] 5

X,T,Θ,*n* ÷ (X,T,Θ,*n* + 1) ,

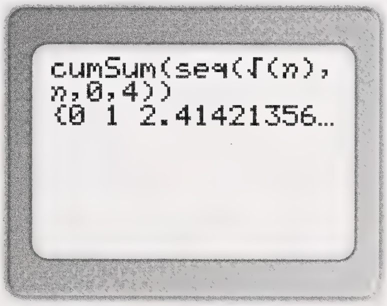
X,T,Θ,*n* , 1 , 5)) ENTER



```
cumSum(seq(n/(n+
1),n,1,5))
{.5 1.166666667...
```

The first five partial sums are $0.5,\ 1.1\overline{6},\ 1.91\overline{6},\ 2.71\overline{6}$, and 3.55.

4. $\sum_{m=0}^{\infty} \sqrt{m}$

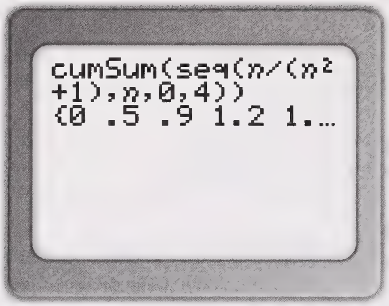


```
cumSum(seq(√(n),
n,0,4))
{0 1 2.41421356...
```

The first five partial sums are 0, 1, 2.414, 4.146, and 6.146.

Enrichment (continued)

6. $\sum_{k=0}^{\infty} \frac{k}{k^2+1}$



```
cumSum(seq(n/(n²
+1),n,0,4))
{0 .5 .9 1.2 1.…
```

The first five partial sums are 0, 0.5, 0.9, 1.2, and 1.435.

b. Textbook questions 7, 8, 9.a., 9.b., 9.c., 10, and 12 of "Applications and Problem Solving," pp. 320 and 321

7. First Approximation

$$\pi = 16\left(\frac{1}{5}\right) - 4\left(\frac{1}{239}\right)$$
$$\doteq 3.183\,263\,598$$

Second Approximation

$$\pi = 16\left(\frac{1}{5} - \frac{1}{3\times5^3}\right) - 4\left(\frac{1}{239} - \frac{1}{3(239)^3}\right)$$
$$\doteq 3.140\,597\,029$$

Third Approximation

$$\pi = 16\left(\frac{1}{5} - \frac{1}{3\times5^3} + \frac{1}{5\times5^5}\right) - 4\left(\frac{1}{239} - \frac{1}{3(239)^3} + \frac{1}{5(239)^5}\right)$$
$$\doteq 3.141\,621\,029$$

Fourth Approximation

$$\pi = 16\left(\frac{1}{5} - \frac{1}{3\times5^3} + \frac{1}{5\times5^5} - \frac{1}{7\times5^7}\right) - 4\left(\frac{1}{239} - \frac{1}{3(239)^3} + \frac{1}{5(239)^5} - \frac{1}{7(239)^7}\right)$$
$$\doteq 3.141\,591\,772$$

Fifth Approximation

$$\pi = 16\left(\frac{1}{5} - \frac{1}{3\times5^3} + \frac{1}{5\times5^5} - \frac{1}{7\times5^7} + \frac{1}{9\times5^9}\right) - 4\left(\frac{1}{239} - \frac{1}{3(239)^3} + \frac{1}{5(239)^5} - \frac{1}{7(239)^7} + \frac{1}{9(239)^9}\right)$$
$$\doteq 3.141\,592\,682$$

Using your calculator,

$$\pi = 3.141\,592\,654\ldots$$

The fifth approximation differs from the actual value of π in the eighth decimal place.

8. **a.**

$$S_1 = 1$$

$$\begin{aligned} S_2 &= S_1 + t_2 \\ &= 1 + \frac{1}{1} \\ &= 2 \end{aligned}$$

$$\begin{aligned} S_3 &= S_2 + t_3 \\ &= 2 + \frac{1}{1 \times 2} \\ &= 2 + \frac{1}{2} \\ &= 2.5 \end{aligned}$$

$$\begin{aligned} S_4 &= S_3 + t_4 \\ &= 2.5 + \frac{1}{1 \times 2 \times 3} \\ &= 2.5 + \frac{1}{6} \\ &= 2.\overline{6} \end{aligned}$$

$$\begin{aligned} S_5 &= S_4 + t_5 \\ &= 2.\overline{6} + \frac{1}{1 \times 2 \times 3 \times 4} \\ &= 2.\overline{6} + \frac{1}{12} \\ &= 2.75 \end{aligned}$$

b. $e \doteq 2.718\,282$

9. **a.** $a : b = (a+b) : a$, where $b = 1$

$$\begin{aligned} \therefore\ \frac{a}{b} &= \frac{a+b}{a} \\ \frac{a}{1} &= \frac{a+1}{a} \\ a^2 &= a+1 \\ a^2 - a - 1 &= 0 \end{aligned}$$

Use the quadratic formula to find a.

$$\begin{aligned} a &= \frac{-(-1) \pm \sqrt{(-1)^2 - 4(1)(-1)}}{2(1)} \\ &= \frac{1 \pm \sqrt{1+4}}{2} \\ &= \frac{1 \pm \sqrt{5}}{2} \end{aligned}$$

$$\therefore\ a = \frac{1+\sqrt{5}}{2} \quad \longleftarrow \text{a cannot be negative.}$$

Enrichment (continued)

b. i. First Approximation

$$\phi = 1$$

Second Approximation

$$\begin{aligned}\phi &= 1 + \frac{1}{1} \\ &= 2\end{aligned}$$

Third Approximation

$$\begin{aligned}\phi &= 1 + \cfrac{1}{1 + \frac{1}{1}} \\ &= 1 + \frac{1}{2} \\ &= 1.5\end{aligned}$$

Fourth Approximation

$$\begin{aligned}\phi &= 1 + \cfrac{1}{1 + \cfrac{1}{1 + \frac{1}{1}}} \\ &= 1 + \cfrac{1}{1 + \frac{1}{2}} \\ &= 1 + \cfrac{1}{\frac{3}{2}} \\ &= 1 + \frac{2}{3} \\ &= 1.\bar{6}\end{aligned}$$

Fifth Approximation

$$\begin{aligned}\phi &= 1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \frac{1}{1}}}} \\ &= 1 + \cfrac{1}{1 + \cfrac{1}{1 + \frac{1}{2}}} \\ &= 1 + \cfrac{1}{1 + \cfrac{1}{\frac{3}{2}}} \\ &= 1 + \cfrac{1}{1 + \frac{2}{3}} \\ &= 1 + \cfrac{1}{\frac{5}{3}} \\ &= 1 + \frac{3}{5} \\ &= 1.6\end{aligned}$$

ii. **First Approximation**

$$\begin{aligned}\phi &= \sqrt{1}\\ &= 1\end{aligned}$$

Second Approximation

$$\begin{aligned}\phi &= \sqrt{1+\sqrt{1}}\\ &= \sqrt{1+1}\\ &= \sqrt{2}\\ &\doteq 1.414\,213\,562\end{aligned}$$

Third Approximation

$$\begin{aligned}\phi &= \sqrt{1+\sqrt{1+\sqrt{1}}}\\ &= \sqrt{1+\sqrt{1+1}}\\ &= \sqrt{1+\sqrt{2}}\\ &\doteq 1.553\,773\,974\end{aligned}$$

Fourth Approximation

$$\begin{aligned}\phi &= \sqrt{1+\sqrt{1+\sqrt{1+\sqrt{1}}}}\\ &= \sqrt{1+\sqrt{1+\sqrt{1+1}}}\\ &= \sqrt{1+\sqrt{1+\sqrt{2}}}\\ &\doteq 1.598\,053\,182\end{aligned}$$

Fifth Approximation

$$\begin{aligned}\phi &= \sqrt{1+\sqrt{1+\sqrt{1+\sqrt{1+\sqrt{1}}}}}\\ &= \sqrt{1+\sqrt{1+\sqrt{1+\sqrt{1+1}}}}\\ &= \sqrt{1+\sqrt{1+\sqrt{1+\sqrt{2}}}}\\ &\doteq 1.611\,847\,754\end{aligned}$$

$$\frac{1+\sqrt{5}}{2} \doteq 1.618\,033\,989$$

The approximations in 9.b.i. agree with 9.a. to 1 decimal place. The approximations in 9.b.ii. agree with 9.a. to 2 decimal places.

c. $$\begin{aligned}\frac{t_2}{t_1} &= \frac{1}{1}\\ &= 1\end{aligned}$$

$$\begin{aligned}\frac{t_3}{t_2} &= \frac{2}{1}\\ &= 2\end{aligned}$$

$$\begin{aligned}\frac{t_4}{t_3} &= \frac{3}{2}\\ &= 1.5\end{aligned}$$

$$\begin{aligned}\frac{t_5}{t_4} &= \frac{5}{3}\\ &= 1.\overline{6}\end{aligned}$$

$$\begin{aligned}\frac{t_6}{t_5} &= \frac{8}{5}\\ &= 1.6\end{aligned}$$

$$\begin{aligned}\frac{t_7}{t_6} &= \frac{13}{8}\\ &= 1.625\end{aligned}$$

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